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Polytopes Associated to Demazure Modules of Symmetrizable Kac–Moody Algebras of Rank Two

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Let ω_1, ω_2 be the two fundamental weights of a symmetrizable Kac–Moody algebra $\mathfrak{g}$ of rank two (hence necessarily affine or finite), and τ an element of the Weyl group. In this paper we construct polytopes $P_\tau(\omega_1), P_\tau(\omega_2) \subset \mathbb{R}^{l(\tau)}$ and a linear map $\xi: \mathbb{R}^{l(\tau)} \to \mathfrak{h}^*$ such that for any dominant weight $\lambda = k_1\omega_1 + k_2\omega_2$, we have $\operatorname{Char} E_\tau(\lambda) = e^\lambda \sum e^{\xi(x)}$, where the sum is over all the integral points x, of the polytope $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. Furthermore, we show that there exists a flat deformation of the Schubert variety S_τ into the toric variety defined by $P_\tau(\omega_1), P_\tau(\omega_2)$.

INTRODUCTION

Much progress has been made in constructing a polytope P, such that the integral points in kP describe the weights of the irreducible representation $V(k\lambda)$ of highest weight $k\lambda$ of a finite or affine Kac–Moody algebra $\mathfrak{g}$. Such polytopes have already been built as the intersection of a set of half-spaces, defined by λ, with a fixed cone. See, for example, [2, 3], where these cones are built using Gelfand–Tsetlin patterns and [18], in which such cones are built (for any Demazure module) using path operators. In [4, 5], we constructed simplicial complexes, with vertices indexed by the Weyl group W, satisfying the above. However, here we would like to build the polytope P as the Minkowski sum of n fixed polytopes, where n is the rank of $\mathfrak{g}$, and we show how this is done when $n = 2$. In [6], we could show a similar result for irreducible representations (and certain Demazure modules) of $\mathfrak{sl}(m)$.

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Since dimension of $V(\sum_{i=}^{n} k_i \omega_i)$, where ω_i are the fundamental weights of $\mathfrak{g}$, is a polynomial in k_i, it seems natural to introduce Minkowski sums: more precisely to construct n fixed polytopes $P_1, \dots, P_n$ in $\mathbb{R}^l$ such that $\mathrm{Card}(\sum_{i=1}^{n} k_i P_i \cap \mathbb{Z}^l) = \dim V(\sum_{i=1}^{n} k_i \omega_i)$. This description does have some advantages with respect to polytopes built previously. For example, our polytopes are closely connected to Standard Monomial Theory (see [14–16]) and Lakshmibai–Seshadri paths (see [17, 19]).

We would like to build P_i, such that its vertices are indexed by the quotient W/W_{ω_i} (where W is the Weyl group of $\mathfrak{g}$ and W_{ω_i} is the stabilizer of ω_i in W). Denoting the vertices by v_σ, $\sigma \in W/W_{\omega_i}$, they shall also have the property that the number of integral points in the polytope P_τ^i, which is the convex envelope of v_σ, $\sigma \preceq \tau$, is equal to the dimension of the Demazure module $E_\tau(\omega_i)$. This is why we believe that our polytopes have potential for generalization to Demazure modules using Minkowski sums of P_τ^i.

Another reason for the use of Minkowski decomposition is due to the belief that there exists a flat deformation of Schubert varieties into toric varieties. Suppose that the Schubert variety S_τ, $\tau \in W$, degenerates into the toric variety X. Then the line bundles $\mathscr{L}_{\omega_1}, \dots, \mathscr{L}_{\omega_n}$ over S_τ deform into n line bundles $\mathscr{L}_1, \dots, \mathscr{L}_n$ over X and we have $\dim H^0(S_\tau, \otimes_{i=1}^{n} \mathscr{L}_i^{\otimes k_i}) = \dim H^0(X, \otimes_{i=1}^{n} \mathscr{L}_i^{\otimes k_i})$ for $k_i \geq 0$. According to [25], the objects $X, \mathscr{L}_1, \dots, \mathscr{L}_n$ define n polytopes $P_\tau^1, \dots, P_\tau^n \subset \mathbb{R}^l$, where $\mathrm{Card}(\sum_{i=1}^{n} k_i P_\tau^i \cap \mathbb{Z}^l) = \dim H^0(X, \otimes_{i=1}^{n} \mathscr{L}_i^{\otimes k_i})$. Since dimension of the Demazure module $E_\tau(\sum_{i=1}^{n} k_i \omega_i)$ is equal to the dimension of $H^0(S_\tau, \otimes_{i=1}^{n} \mathscr{L}_{\omega_i}^{\otimes k_i})$ (see [1, 21] for $\mathfrak{g}$ finite dimensional and [13, 22] for $\mathfrak{g}$ infinite dimensional), the necessity of using Minkowski decomposition, if such a flat deformation exists, becomes evident.

In this paper we shall consider only symmetrizable Kac–Moody algebras $\mathfrak{g}$, of rank two (these algebras are necessarily finite or affine since the rank of the Cartan matrix can be only two or one). We will construct polytopes $P_\tau^1, P_\tau^2 \subset \mathbb{R}^{l(\tau)}$ such that the number of integral points in $k_1 P_\tau^1 + k_2 P_\tau^2$ is equal to the dimension of $E_\tau(k_1 \omega_1 + k_2 \omega_2)$. We then show that the Schubert variety S_τ deforms into the toric variety defined by the pair P_τ^1, P_τ^2. To do this, we present a plan in a general setting (that is, $\mathfrak{g}$ of arbitrary rank), and we show that the plan works when $\mathfrak{g}$ is of rank two.

1. CONJECTURES FOR KAC–MOODY ALGEBRAS OF FINITE OR AFFINE TYPE

Let $\mathfrak{g}$ be a finite or affine Kac–Moody algebra of rank n, $\mathfrak{b}$ a Borel subalgebra, and $\mathfrak{h}$ a Cartan subalgebra in $\mathfrak{b}$. To every dominant weight λ we associate an irreducible representation $V(\lambda)$, with $v_\lambda \in V(\lambda)$ the eigenvector of weight λ. Let τ be an element of the Weyl group W. Then $E_\tau(\lambda)$ denotes the subspace $U(\tau \mathfrak{b} \tau^{-1})v_\lambda$ called the Demazure module. Finally, W_{ω_i} is the stabilizer of the fundamental weight ω_i in W.

Fix an element $\tau \in W$. Using the combinatorics of Lakshmibai–Seshadri (LS) paths (recently defined by Littelmann in [17], I shall first define a polytope $P_\tau(\omega_i)$, such that the number of integral points in $P_\tau(\omega_i)$ is equal to dim $E_\tau(\omega_i)$. This combinatorics implies that μ is a weight of $E_\tau(\omega_i)$ if and only if there exists a chain of elements in W/W_{ω_i}: $\tau \succeq \tau_1 \succ \cdots \succ \tau_r$ and a sequence of positive rationals $b_1, \dots, b_r$ with $\sum_{i=1}^r b_i = 1$ such that $\mu = b_1\tau_1(\omega_i) + \cdots + b_r\tau_r(\omega_i)$, where the denominators of b_i satisfy certain integrality conditions depending on τ_i and ω_i. This property leads us to the following problem.

Problem 1. *Let $\bar{\tau}$ be the image of τ in W/W_{ω_i}. Construct a polytope with integral vertices $\tilde{P}_\tau(\omega_i)$ in $\mathbb{R}^{l_i}$ such that*

(i) *The vertices of $\tilde{P}_\tau(\omega_i)$ are indexed by the set $\{w \preceq \bar{\tau}\}$. Moreover, the convex envelope of the vertices $v_{\tau_1}, \dots, v_{\tau_r}$, where $\tau_1 \succ \cdots \succ \tau_r$, is a simplex and $\tilde{P}_\tau(\omega_i)$ is triangulized by the set of such simplexes. In particular, every point x of $\tilde{P}_\tau(\omega_i)$ is written as $x = \sum_{i=1}^r b_i(v_{\tau_i})$, where $\tau_1 \succ \cdots \succ \tau_r$ and $\sum_{i=1}^r b_i = 1$.*

(ii) *Let $k \geq 1$. A point $\sum_{i=1}^r b_i(kv_{\tau_i})$ of $k \cdot \tilde{P}_\tau(\omega_i)$ has integral coordinates if and only if $\mu = b_1\tau_1(k\omega_1) + \cdots + b_r\tau_r(k\omega_r)$ is a weight of $E_\tau(k\omega_i)$.*

In [6], we could construct such polytopes for $\mathfrak{sl}(n)$.

Let $\lambda = \sum_{i=1}^n k_i\omega_i$ be a dominant weight. By algebraic geometry arguments, we already know that dim $E_\tau(\lambda)$ is a polynomial in variables k_i. This suggests to us to use Minkowski sums as follows: Let $P_1, \dots, P_n$ be convex polytopes in $\mathbb{R}^l$. Suppose that we can write an integral point in the Minkowski sum $\sum_{i=1}^n k_i P_i$ as

$$\text{an integral point of } \sum_{i=1}^n k_i P_i = k_i \text{ integral points of } P_1 + \cdots + k_n \text{ integral points of } P_n. \quad (1.1)$$

The Ehrhart theorem [8] implies that the number of integral points in $\sum_{i=1}^n k_i P_i$ is a polynomial in the variables k_i. Therefore, what we would like to do is the following.

Problem 2. *Find an embedding ψ_i: $\tilde{P}_\tau(\omega_i) \hookrightarrow \mathbb{R}^{l(\tau)}$ such that*

(i) *the polytopes $P_\tau(\omega_i) = \psi_i(\tilde{P}_\tau(\omega_i))$ satisfy property* (1.1).

(ii) *the number of integral points of $\sum_{i=1}^n k_i P_\tau(\omega_i)$ is equal to* dim $E_\tau(\sum_{i=1}^n k_i\omega_i)$.

Up to now, we could find such imbeddings for certain Demazure modules of $\mathfrak{sl}(n)$ (see [6]).

Let now S_τ be the Schubert variety associated to τ and $\mathscr{A}_\tau$ the graded algebra $\oplus_{k_1,\ldots,k_n} H^0(S_\tau, \otimes_{i=1}^n \mathscr{L}_{\omega_i}^{\otimes k_i})$. The algebra $\mathscr{A}_\tau$ is the homogeneous coordinate ring of the multicone over S_τ. That is the $B\tau B$-orbit of $\oplus_{i=1}^n \mathbb{C} v_{\omega_i}$ in $\oplus_{i=1}^n V(\omega_i)$, where B is a Borel subgroup of G. In [12], it has been shown that the canonical homomorphism $\oplus_{k_1,\ldots,k_n} \otimes_{i=1}^n \operatorname{Sym}^{k_i} H^0(S_\tau, \mathscr{L}_{\omega_i}) \to \mathscr{A}_\tau$ is surjective and that the kernel I_τ is generated by quadratic relations. In other words, $\mathscr{A}_\tau = \mathbb{C}[x_i^{v_i}]/I_\tau$, where $i = 1, \ldots, n$ and $\{v_i\}$ is a basis of eigenvectors of $E_\tau(\omega_i)$.

On the other hand, recall that Problem 2 provides us with n convex polytopes $P_\tau(\omega_1), \ldots, P_\tau(\omega_n)$ whose vertices have integral coordinates. Set $\mathscr{B}_{k_1,\ldots,k_n}$ to be the vector space over $\mathbb{C}$ generated by $\{x^\alpha\}$, where α is an integral point of $\sum_{i=1}^n k_i P_\tau(\omega_i)$. Then $\mathscr{B}_\tau := \oplus_{k_1,\ldots,k_n} \mathscr{B}_{k_1,\ldots,k_n}$ is a graded algebra where the multiplication is defined by $x^\alpha x^\beta = x^{\alpha+\beta}$. According to [25], there exists a toric variety X_τ equipped with fiber bundles $\mathscr{L}_1, \ldots, \mathscr{L}_n$ such that $H^0(X_\tau, \otimes_{i=1}^n \mathscr{L}_i^{\otimes k_i}) \simeq \mathscr{B}_\tau$.

We have a canonical homomorphism $\oplus_{k_1,\ldots,k_n} \otimes_{i=1}^n \operatorname{Sym}^{k_i} H^0(X_\tau, \mathscr{L}_i) \to \mathscr{B}_\tau$. Recall that the polytopes $P_\tau(\omega_i)$ satisfy property 1.1. This property implies

1. the homomorphism $\oplus_{k_1,\ldots,k_n} \otimes_{i=1}^n \operatorname{Sym}^{k_i} H^0(X_\tau, \mathscr{L}_i) \to \mathscr{B}_\tau$ is surjective.
2. the kernel J_τ of this homomorphism is generated by quadratic relations.

In other words, $\mathscr{B}_\tau = \mathbb{C}[x_i^{p_i}]/J_\tau$, where $i = 1, \ldots, n$ and $\{p_i\}$ is the set of all points with integral coordinates in $P_\tau(\omega_i)$. Since property (ii) of Problem 1 sets up a natural one-to-one correspondence between the integral points of $P_\tau(\omega_i)$ and a basis of eigenvectors of $E_\tau(\omega_i)$, we have $\mathscr{B}_\tau = \mathbb{C}[x_i^{v_i}]/J_\tau$, where $\{v_i\}$ is, as before, a basis of eigenvectors of $E_\tau(\omega_i)$. We hope to show the following.

PROBLEM 3. *Let S_τ be a Schubert variety. There exists a flat family over $\mathbb{C}[t]$ of quotients of the graded algebra $\mathbb{C}[x_i^{v_i}]$ (where $i = 1, \ldots, n$ and $\{v_i\}$ is a basis of eigenvectors of $E_\tau(\omega_i)$) whose fiber over 0 is the homogeneous coordinate ring of the multicone over the toric variety X_τ and whose fiber over any $(t-u)$, for $0 \neq u \in \mathbb{C}$, is the homogeneous coordinate ring of the multicone over S_τ.*

We would like to show this using theorem 15.17 of [7]. That is, we want to find an integral weight function ϑ on $\mathbb{C}[x_i^{v_i}]$ such that $in_\vartheta(I_\tau) \simeq J_\tau$.

We shall solve Problems 1 and 2 for symmetrizable Kac–Moody algebras of rank two. That is $\mathfrak{sl}(3)$, $\mathfrak{sp}(4)$, $\mathfrak{g}_2$ (finite dimensional) and $A_1^{(1)}$, $A_2^{(2)}$

(infinite dimensional). Problem 3 is solved only for finite-dimensional Lie algebras of rank two.

The sections are organized as follows. In Section 2, we shall present three theorems. Theorem 2.2 explains how to construct polytopes satisfying Problem 1 for fundamental weights ω_1, ω_2. This theorem is proved in Section 3, where we also give some examples.

Theorem 2.4 explains how to imbed the polytopes constructed in Theorem 2.2 in $\mathbb{R}^{l(\tau)}$. We will sketch out the proof of this theorem in Section 4 and show all the details for $\mathfrak{g}_2$, τ maximal and for $A_1^{(1)}$, τ arbitrary in the Appendix.

The proof of Theorem 2.5 is given in Section 5. We prove this theorem for the most difficult case: g_2, τ maximal. We shall do this by finding explicitly the relations which define G_2/B (where B is a Borel subgroup) and show that they deform into the relations defining the corresponding toric variety.

Finally, in Section 6, we shall give an application of Theorem 2.4. We construct a linear map $f\colon \mathbb{R}^{l(\tau)} \to \mathfrak{h}^*$ such that for the dominant weight $\lambda = k_1\omega_1 + k_2\omega_2$, we have Char $E_\tau(\lambda) = e^\lambda \sum e^{f(x)}$, where the sum is over the integral points of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$.

2. THEOREMS FOR RANK-TWO LIE ALGEBRAS

From now on, we assume that $\mathfrak{g}$ is of rank two. For $\tau \in W$ and $\lambda \in X^+$, we write $\Pi_\tau(\lambda)$ to be the set of LS paths starting in the direction $w(\lambda)$ with $w \preceq \tau$ (see [17, 19]). We denote by s_{α_i} the simple reflection with respect to the simple root α_i. Let ω_i, where $i = 1, 2$, be a fundamental weight of $\mathfrak{g}$. We recall that W/W_{ω_i} can be identified with the subset of W consisting of elements $w = s_{\alpha_{i_l}} \cdots s_{\alpha_{i_1}}$ (this is a reduced expression) such that $s_{\alpha_{i_1}} = s_{\alpha_i}$ and l is strictly less than the length of the maximal element in W. Moreover, for an element $\tau \in W$, the set $[1, \bar{\tau}] = \{w \in W/W_{\omega_i} | \bar{\tau} \succeq w\}$, where $\bar{\tau}$ is the image of τ in W/W_{ω_i}, is totally ordered. That is, $[1, \bar{\tau}] = \{\bar{\tau} = \tau_l \succ \cdots \succ \tau_0 = 1\}$.

Definition 2.1. Let $\tau \in W$ and $\bar{\tau} \in W/W_{\omega_i}$ its image, with $[1, \bar{\tau}] = \{\bar{\tau} = \tau_l \succ \cdots \succ 1 = \tau_0\}$. Denote by α_j the positive roots such that $\tau_j = s_{\alpha_j}\tau_{j-1}$. Set

$$v_{\tau_j} = \sum_{r=1}^{j} \langle \tau_r(\lambda), \alpha_r^\vee \rangle e_r, \tag{2.1}$$

where $\{e_r\}_{r=1}^l$ is the canonical basis of $\mathbb{R}^l$. Define $\tilde{P}_\tau(\omega_i)$ to be the convex envelope of the points $v_{\tau_0} = 0, v_{\tau_1}, \ldots, v_{\tau_l}$.

THEOREM 2.2. *The vertices of* $\tilde{P}_\tau(\omega_i)$ *have integral coordinates and are indexed by the set* $[1, \bar{\tau}]$. *Moreover, there is a canonical bijection between integral points in* $k\tilde{P}_\tau(\omega_i)$ *and Lakshmibai–Seshadri paths in* $\Pi_\tau(k\omega_i)$.

We shall give the proof of this theorem in Section 3, where we will also specify the $\tilde{P}_\tau(\omega_i)$ explicitly for $\mathfrak{g}$ of rank two and arbitrary τ.

Suppose $\tau = s_{\alpha_{i_l}} \cdots s_{\alpha_{i_1}} \in W$ is a reduced expression of τ (then $i_k = i_1$ if k is odd and $i_k = i_2$ if k is even). We define imbeddings ψ_i: $\tilde{P}_\tau(\omega_i) \hookrightarrow \mathbb{R}^l$ as

$$\psi_{i_1}(x) = \begin{cases} x & \text{if } \tau \text{ is not the maximal element in } W, \\ (0, x) & \text{otherwise,} \end{cases} \tag{2.2}$$

and

$$\psi_{i_2}(x) = (x, 0). \tag{2.3}$$

DEFINITION 2.3. Define $P_\tau(\omega_i) := \psi_i(\tilde{P}_\tau(\omega_i))$.

THEOREM 2.4. *The polytopes* $P_\tau(\omega_1)$ *and* $P_\tau(\omega_2)$ *are such that*

(i) *they satisfy property* (1.1).

(ii) *there is a canonical bijection between integral points of the Minkowski sum* $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$ *and Lakshmibai–Seshadri paths in* $\Pi_\tau(k_1\omega_1 + k_2\omega_2)$.

The proof of this theorem is briefly sketched out in Section 4. We shall give a detailed proof for $\mathfrak{g}_2$ when τ is the maximal element, and for the affine Lie algebra $A_1^{(1)}$ when τ is arbitrary, in Sections A.1 and A.2, respectively.

THEOREM 2.5. *The Schubert variety* S_τ *equipped with the line bundles* $\mathscr{L}_{\omega_1}$ *and* $\mathscr{L}_{\omega_2}$ *degenerates into the toric variety* X_τ *equipped with the line bundles defined by* $P_\tau(\omega_1)$ *and* $P_\tau(\omega_2)$.

We can only prove this theorem for finite-dimensional $\mathfrak{g}$ (that is, for $\mathfrak{sl}(3)$, $\mathfrak{sp}(4)$, and $\mathfrak{g}_2$). We can also show that the deformation of G/B (G of rank two and B a Borel subgroup in G) is compatible with the deformation of the Schubert varieties S_τ in G/B.

3. PROOF OF THEOREM 2.2 AND EXAMPLES

Proof of Theorem 2.2. The assertion that the vertices of $\tilde{P}_\tau(\omega_i)$ are indexed by $[1, \bar{\tau}]$ is clear from Definition 2.1. For the second part, we shall give the bijection between integral points of $k\tilde{P}_\tau(\omega_i)$ and Lakshmibai–

Seshadri paths in $\Pi_\tau(k\omega_i)$, but we refer the reader to section four of [4] for further details.

Let x be an integral point in $k\tilde{P}_\tau(\omega_i)$. We can write $x = \sum_{j=0}^{l} k(b_j v_{\tau_j})$, where $b_j \geq 0$ and $\sum b_j = 1$. Define

$$x = \sum_{j=0}^{l} k\big(b_j v_{\tau_j}\big) \xrightarrow{\phi_i} (\tau_l \succ \tau_{l-1} \succ \cdots \succ \tau_0;$$

$$0 \leq b_l \leq b_l + b_{l-1} \leq \cdots \leq (b_l + \cdots + b_0)). \quad (3.1)$$

This application is a bijection between integral points of $k\tilde{P}_\tau(\omega_i)$ and Lakshmibai–Seshadri paths in $\Pi_\tau(k\omega_i)$. ∎

3.1. *$\tilde{P}_\tau(\omega_1)$ and $\tilde{P}_\tau(\omega_2)$ for $\mathfrak{sl}(3)$*

Let α_1 and α_2 be the two simple roots of $\mathfrak{sl}(3)$. Recall that $\langle \alpha_i, \alpha_j^\vee \rangle$ is 2 when $i = j$ and -1 when $i \neq j$. For simplicity, we denote by s_i the reflection s_{α_i}. With this notation, $W/W_{\omega_1} = \{\tau_2 = s_2 s_1 \succ \tau_1 = s_1 \succ \tau_0 = 1\}$. According to Eq. (2.1), we then have

$$v_{\tau_1} = (0,0), \qquad v_{\tau_2} = (0,1), \qquad v_{\tau_3} = (1,1). \quad (3.2)$$

Then the simplex $\tilde{P}_\tau(\omega_1)$ is the convex envelope of $\{v_{\tau_k}\}_{\tau_k \preceq \bar{\tau}}$. Note that $\tilde{P}_\tau(\omega_i) \subset \mathbb{R}^{l(\bar{\tau})}$. Due to the symmetry between α_1 and α_2, the other case is similar.

3.2. *$\tilde{P}_\tau(\omega_1)$ and $\tilde{P}_\tau(\omega_2)$ for $\mathfrak{sp}(4)$*

Let α_1 and α_2 be the two simple roots of $\mathfrak{sp}(4)$, where α_1 is the shorter root. Then $\langle \alpha_i, \alpha_i^\vee \rangle = 2$, $\langle \alpha_1, \alpha_2^\vee \rangle = -1$, and $\langle \alpha_2, \alpha_1^\vee \rangle = -2$. For simplicity, we write $s_i = s_{\alpha_i}$. The set $W/W_{\omega_1} = \{\tau_3 = s_1 s_2 s_1 \succ \tau_2 = s_2 s_1 \succ \tau_1 = s_1 \succ \tau_0 = 1\}$. According to Eq. (2.1), we have

$$v_{\tau_0} = (0,0,0), \qquad v_{\tau_1} = (0,0,1), \qquad v_{\tau_2} = (0,1,1), \qquad v_{\tau_3} = (1,1,1). \quad (3.3)$$

And so the simplex $\tilde{P}_\tau(\omega_1)$ is the convex envelope of $\{v_{\tau_k}\}_{\tau_k \preceq \bar{\tau}}$. On the other hand, $W/W_{\omega_2} = \{\sigma_3 = s_2 s_1 s_2 \succ \sigma_2 = s_1 s_2 \succ \sigma_1 = s_2 \succ \sigma_0 = 1\}$. Equation (2.1) gives us

$$v_{\sigma_0} = (0,0,0), \qquad v_{\sigma_1} = (0,0,1), \qquad v_{\sigma_2} = (0,2,1), \qquad v_{\sigma_3} = (1,2,1). \quad (3.4)$$

And the simplex $\tilde{P}_\tau(\omega_2)$ is the convex envelope of $\{v_{\sigma_k}\}_{\sigma_k \preceq \bar{\tau}}$.

3.3. $\tilde{P}_\tau(\omega_1)$ and $\tilde{P}_\tau(\omega_2)$ for $\mathfrak{g}_2$

Let α_1 and α_2 be the two simple roots of $\mathfrak{g}_2$, where α_1 is the shorter root. We have $\langle \alpha_i, \alpha_i^\vee \rangle = 2$, $\langle \alpha_1, \alpha_2^\vee \rangle = -1$, $\langle \alpha_2, \alpha_1^\vee \rangle = -3$. As before, for simplicity, we write $s_i = s_{\alpha_i}$. The set $W/W_{\omega_1} = \{\tau_5 = s_1s_2s_1s_2s_1 \succ \tau_4 = s_2s_1s_2s_1 \succ \tau_3 = s_1s_2s_1 \succ \tau_2 = s_2s_1 \succ \tau_1 = s_1 \succ \tau_0 = 1\}$. From Eq. (2.1) we get

$$\begin{aligned} v_{\tau_0} &= (0,0,0,0,0), & v_{\tau_1} &= (0,0,0,0,1), & v_{\tau_2} &= (0,0,0,1,1), \\ v_{\tau_3} &= (0,0,2,1,1), & v_{\tau_4} &= (0,1,2,1,1), & v_{\tau_5} &= (1,1,2,1,1). \end{aligned} \tag{3.5}$$

So the simplex $\tilde{P}_\tau(\omega_i)$ is the convex envelope of the points $\{v_{\tau_k}\}_{\tau_k \preceq \bar{\tau}}$. On the other hand, the set $W/W_{\omega_2} = \{\sigma_5 = s_2s_1s_2s_1s_2 \succ \sigma_4 = s_1s_2s_1s_2 \succ \sigma_3 = s_2s_1s_2 \succ \sigma_2 = s_1s_2 \succ \sigma_1 = s_2 \succ \sigma_0 = 1\}$ and so

$$\begin{aligned} v_{\sigma_0} &= (0,0,0,0,0), & v_{\sigma_1} &= (0,0,0,0,1), & v_{\sigma_2} &= (0,0,0,3,1), \\ v_{\sigma_3} &= (0,0,2,3,1), & v_{\sigma_4} &= (0,3,2,3,1), & v_{\sigma_5} &= (1,3,2,3,1). \end{aligned} \tag{3.6}$$

Therefore, $\tilde{P}_\tau(\omega_2)$ is the convex envelope of the points $\{v_{\sigma_k}\}_{\sigma_k \preceq \bar{\tau}}$.

3.4. $\tilde{P}_\tau(\omega_1)$ and $\tilde{P}_\tau(\omega_2)$ for $A_1^{(1)}$

Let α_1 and α_2 be the two simple roots of the affine Lie algebra $A_1^{(1)}$. We have $\langle \alpha_i, \alpha_i^\vee \rangle = 2$ and $\langle \alpha_i, \alpha_j^\vee \rangle = -2$ when $i \neq j$. For simplicity, let $s_i = s_{\alpha_i}$. Consider $\bar{\tau} = s_{i_l} \cdots s_{i_1} \in W/W_{\omega_1}$. Then $i_k = 1$ if i_k is odd and 2 if i_k is pair. Moreover, $[1, \bar{\tau}] = \{\tau_k \mid \tau_0 = 1$ and $\tau_k = s_{i_k} \cdots s_{i_1}$, where $1 \leq k \leq l\}$.
Hence, according to Eq. (2.1), $\tilde{P}_\tau(\omega_1) \subset \mathbb{R}^l$ is the convex envelope of

$$v_{\tau_k} = \sum_{i=1}^{k} ie_i. \tag{3.7}$$

Due to the symmetry between α_1 and α_2, the other case is similar.

3.5. $\tilde{P}_\tau(\omega_1)$ and $\tilde{P}_\tau(\omega_2)$ for $A_2^{(2)}$

Let α_1 and α_2 be the two simple roots of $A_2^{(2)}$ with α_1 the shorter root. We have $\langle \alpha_i, \alpha_i^\vee \rangle = 2$, $\langle \alpha_1, \alpha_2^\vee \rangle = -1$, and $\langle \alpha_2, \alpha_1^\vee \rangle = -4$. Consider $\bar{\tau} = s_{i_l} \ldots s_{i_1} \in W/W_{\omega_1}$. Then $i_k = 1$ if i_k is odd and 2 if it is even. We have $[1, \bar{\tau}] = \{\tau_k \mid \tau_0 = 1$ and $\tau_k = s_{i_k} \ldots s_{i_1}$, where $1 \leq k \leq l\}$. Then $\tilde{P}_\tau(\omega_1) \subset \mathbb{R}^l$ has vertices

$$v_{\tau_k} = \sum_{2i+1 \leq k} (2i+1) e_{2i+1} + \sum_{2i \leq k} ie_{2i}. \tag{3.8}$$

On the other hand, suppose $\bar{\tau} = s_{i_l} \dots s_{i_1}$ is an element of W/W_{ω_2}. Then we can assume that $i_k = 2$ if k is odd and 1 if k is even. The set $[1, \bar{\tau}] = \{\tau_k \mid \tau_0 = 1 \text{ and } \tau_k = s_{i_k} \dots s_{i_1}, \text{ where } 1 \leq k \leq l\}$. Therefore, according to Eq. (2.1), $\tilde{P}_\tau(\omega_2)$ is in $\mathbb{R}^l$ and its vertices are

$$v_{\tau_k} = \sum_{2i+1 \leq k} (2i+1) e_{2i+1} + \sum_{2i \leq k} 4ie_{2i}. \tag{3.9}$$

The dimensions of Demazure modules for $A_1^{(1)}$ and $A_2^{(2)}$ have been calculated in [23].

4. PROOF OF THEOREM 2.4 AND EXAMPLES

We shall first show that an integral point in $kP_\tau(\omega_i)$ can be written as the sum of k integral points in $P_\tau(\omega_i)$.

LEMMA 4.1. *Let $P \subset \mathbb{R}^l$ be a polytope with vertices $v_0 = 0, v_1, \dots, v_r$ which are*

$$v_j = \sum_{r=1}^{j} a_r e_r, \quad \textit{where } a_r \in \mathbb{N} \textit{ for } r, \tag{4.1}$$

and $\{e_r\}_{r=1}^l$ is the canonical basis of $\mathbb{R}^l$. Then an integral point in $k \cdot P$ can be written as the sum of k integral points of P.

Proof. The proof is by induction on k. The assertion is clear when $k = 1$. So now let $k > 1$. Suppose x is an integral point in $k \cdot P$. Then $x = \sum_{j=0}^l b_j v_j$, where $b_j \geq 0$ and $\sum_{j=0}^l b_j = k$. Observe that the e_r-coordinate of x is $a_r(\sum_{i=r}^l b_i)$.

Let t be the smallest number such that $\sum_{i=t}^l b_i < 1$. Set

$$\begin{aligned} y &= b_l v_l + \cdots + b_t v_t + \left(1 - \textstyle\sum_{i=t}^l b_i\right) v_{t-1} \\ &= \sum_{j=t}^{l} b_j \left(\sum_{r=1}^{j} a_r e_r \right) + \left(1 - \sum_{i=t}^{l} b_i \right) \left(\sum_{r=1}^{t-1} a_r e_r \right) \\ &= \sum_{r=t}^{l} \left(a_r \sum_{i=r}^{l} b_i \right) e_r + \sum_{r=1}^{t-1} a_r e_r. \end{aligned} \tag{4.2}$$

Note that y is a point in P. Moreover, for $r = t, \dots, l$, the e_r-coordinate of x and y is the same, and so this coordinate is a positive integer. On the other hand, for $r = 1, \dots, t-1$, the e_r-coordinate of y is a_r, which is also a positive integer. Hence, y is a point of P with integral coordinates. We have

$$x = y + z, \quad \text{where } z = \left(\sum_{i=t-1}^{l} b_i - 1 \right) v_{t-1} + b_{t-2} v_{t-2} + \cdots + b_0 v_0. \tag{4.3}$$

Since x and y have integral coordinates, then so does z. Moreover, $(\sum_{i=t-1}^{l} b_i) - 1 + \sum_{i=0}^{t-2} b_i = k - 1$. Therefore, z is an integral point of $(k-1) \cdot P$. Now the assertion follows by induction. ∎

COROLLARY 4.2. *Let $P_\tau(\omega_i)$ be the polytope defined in Eqs.* (2.2) *and* (2.3). *Then we can write a point in $kP_\tau(\omega_i)$ as a sum of k integral points in $P_\tau(\omega_i)$.*

Proof. According to Definition 2.1 and Definition 2.3, the vertices of $P_\tau(\omega_i)$ satisfy the hypothesis of Lemma 4.1. Hence the result.

PROPOSITION 4.3. *Let $\tau = s_{\alpha_{i_l}} \dots s_{\alpha_{i_2}} s_{\alpha_{i_1}}$ be a reduced expression for τ. Suppose x is an integral point of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. Then there exists an element $\eta \in W$ such that we can write $x = x_1 + x_2$, where $x_1 = \sum_{\{\kappa \in W/W_{\omega_{i_1}} | 1 \preceq \kappa \preceq \overline{\eta}\}} a_\kappa v_\kappa$ is an integral point of $k_{i_1} P_\tau(\omega_{i_1})$ and $x_2 = \sum_{\{\sigma \in W/W_{\omega_{i_2}} | \overline{\eta} \preceq \sigma \preceq \overline{\tau}\}} b_\sigma v_\sigma$ is an integral point of $k_{i_2} P_\tau(\omega_{i_2})$.*

Proof. We will briefly sketch out the proof here. However, in Sections A.1 and A.2 of the Appendix, we shall show all the details for $\mathfrak{g}_2$ and $A_1^{(1)}$, respectively.

We denote by κ_i (respectively, σ_i) the element of length i in W/W_{i_1} (respectively, in W/W_{i_2}). Moreover, let l_1 (respectively, l_2) be such that $\overline{\tau} = \kappa_{l_1}$ in W/W_{i_1} (respectively, $\overline{\tau} = \sigma_{l_2}$ in W/W_{i_2}). Note that $l_2 = l_1$ if τ is maximal and $l_2 = l_1 - 1$ otherwise. Therefore, the vertices of $P_\tau(\omega_{i_1})$ are $v_{\kappa_0}, \dots, v_{\kappa_{l_1}}$ and the vertices of $P_\tau(\omega_{i_2})$ are $v_{\sigma_0}, \dots, v_{\sigma_{l_2}}$.

For simplicity of notation set $v_{\kappa_i} = v_i$ and $v_{\sigma_i} = w_i$. First note that there exist numbers $r_j^i \geq 0$ such that we have

$$v_j = \sum_{i=1}^{j-1} r_j^i w_i + v_1. \tag{4.4}$$

The existence of such numbers is left to the reader. We have calculated them for $\mathfrak{g}_2$ in Eq. (A.3) and for $A_1^{(1)}$ in Eq. (A.4). We note, however, that r_j^i have the following properties:

1. For all $j \geq i+2$, we have $r_j^i = r_{i+2}^i$. Denote $r_{i+2}^i := r^i$.
2. For all i and all $j \geq i+2$, we have $r_{i+1}^i = r^i + \cdots + r^{j-2} + r_j^{j-1}$.

Since $x \in k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$, then $x = \sum_{i=0}^{l_1} a_i v_i + \sum_{i=0}^{l_2} b_i w_i$, where $a_i \geq 0$, $b_i \geq 0$, and $\sum_{i \leq l_1} a_i = k_{i_1}$, $\sum_{i \leq l_2} b_i = k_{i_2}$. From Eq. (4.4) we get

$$x = \sum_{i=1}^{l_2} c_i w_i + b_0 w_0 + \left(\sum_{j=1}^{l_1} a_j \right) v_1 + a_0 v_0,$$

$$\text{where } c_i = b_i + r_{i+1}^i a_{i+1} + r^i \left(\sum_{j=i+2}^{l_1} a_j \right). \tag{4.5}$$

Using the aforementioned properties of r_j^i, we obtain $\Sigma_{i=p}^{l_2} c_i = \Sigma_{i=p}^{l_2} b_i + r_{p+1}^p(\Sigma_{i=p+1}^{l_1} a_i)$. Now, let t be the smallest number such that $\Sigma_{i=t}^{l_2} c_i \leq k_{i_2}$. We then have

$$x = \sum_{i=t}^{l_2} c_i w_i + \left(k_{i_2} - \sum_{i=t}^{l_2} c_i \right) w_{t-1} + \left(\sum_{i=t-1}^{l_2} c_i - k_{i_2} \right) w_{t-1} + \sum_{i=1}^{t-2} c_i w_i + \left(\sum_{j=1}^{l_1} a_j \right) v_1 + a_0 v_0. \tag{4.6}$$

Let $d_t = 1/r_t^{t-1}(\Sigma_{i=t-1}^{l_2} c_i - k_{i_2})$ and define d_j, $1 \leq j \leq t$, by induction as follows:

$$d_j = \frac{1}{r_j^{j-1}} \left(c_{j-1} - r^{j-1} \sum_{i=j+1}^{t} d_i \right) \quad \text{and} \quad d_1 = \sum_{j=1}^{l_1} a_j - \sum_{j=2}^{t} d_j. \tag{4.7}$$

Using $\Sigma_{i=p}^{l_2} c_i = \Sigma_{i=p}^{l_2} b_i + r_{p+1}^p(\Sigma_{i=p+1}^{l_1} a_i)$, by induction one can prove

$$d_j + \cdots + d_t = \sum_{i=j}^{l_1} a_i + \frac{1}{r_j^{j-1}} \left(\sum_{j=1}^{l_2} b_j - k_{i_2} \right) \quad \text{for } 2 \leq j \leq t, \tag{4.8}$$

so that

$$d_j = \begin{cases} \left(1/r_t^{t-1}\right)\left(\Sigma_{i=t-1}^{l_2} b_i - k_{i_2}\right)\Sigma_{i=t}^{l_1} a_i & \text{if } j = t, \\ \left(1/r_j^{j-1}\right) b_{j-1} + a_j + \left(r^{j-1}/r_j^{j-1} r_{j+1}^j\right)\left(k_{i_2} - \Sigma_{i=j}^{l_2} b_i\right) & \text{if } 2 \leq j < t, \\ a_1 + \left(1/r_2^1\right) b_0 & \text{if } j = 1. \end{cases} \tag{4.9}$$

It follows that $d_j \geq 0$. Moreover, by definition of d_1, the sum $\Sigma_{j=1}^t d_j + a_0 = k_{i_1}$. Finally, we have

$$x = \underbrace{\sum_{i=t}^{l_2} c_i w_i + \left(k_{i_2} - \sum_{i=t}^{l_2} c_i \right) w_{t-1}}_{x_2} + \underbrace{\sum_{i=1}^{t} d_i v_i + a_0 v_0}_{x_1}. \tag{4.10}$$

A closer look at the vertices w_i and v_j reveals that $c_i w_i$, $t \leq i \leq l_2$, and $(k_{i_2} - \Sigma_{i=t}^{l_2} c_i) w_{t-1}$, and $d_j v_j$, $1 \leq j \leq t$, and $a_0 v_0$ are points with integral coordinates. That is, x_1 is an integral point of $k_{i_1} P_\tau(\omega_{i_1})$ and x_2 is an integral point of $k_{i_2} P_\tau(\omega_{i_2})$. Hence, we can write x as claimed, where $\eta = s_{i_{t+1}} \ldots s_{i_2} s_{i_1}$. ∎

Let $\lambda = k_1\omega_1 + k_2\omega_2$. Using Proposition 4.3 and theorem 10.1 of [20], we can give a natural bijection between integral points in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$ and LS paths in $\Pi_\tau(\lambda)$.

Keep the notation as in the proof of Proposition 4.3. Suppose $\pi \in \Pi_\tau(\lambda)$. Then there exist positive integers e_i, $i = 1, \ldots, l$, such that $\pi = f_{\alpha_{i_l}}^{e_l} \ldots f_{\alpha_{i_1}}^{e_3} f_{\alpha_{i_2}}^{e_2} f_{\alpha_{i_1}}^{e_1}(\pi_\lambda)$, where π_λ is the linear path from 0 to λ (see, for example, [17]). Let

$$f_{\alpha_{i_l}}^{e_l} \ldots f_{\alpha_{i_1}}^{e_3} f_{\alpha_{i_2}}^{e_2} f_{\alpha_{i_1}}^{e_1}(\pi_{k_2\omega_2} * \pi_{k_1\omega_1}) = \pi_2 * \pi_1, \quad \text{where } \pi_j \in \Pi_\tau\big(k_{i_j}\omega_{i_j}\big). \tag{4.11}$$

According to theorem 10.1 of [20], $\pi_2 = (\sigma_{l_2} \succ \cdots \succ \sigma_{t-1};\ 0 = \beta_l \le \beta_{l-1} \le \cdots \le \beta_{t-1} = 1)$, where $\sigma_j = s_{\alpha_{i_j}} \ldots s_{\alpha_{i_2}} s_{\alpha_{i_1}} s_{\alpha_{i_2}}$ and $\pi_1 = (\tau_t \succ \cdots \succ \tau_0;\ 0 = \alpha_{t+1} \le \alpha_t \le \cdots \le \alpha_0 = 1)$, where $\tau_j = s_{\alpha_{i_j}} \ldots s_{\alpha_l} s_{\alpha_{i_2}} s_{\alpha_{i_1}}$. From Eq. (3.1), we obtain $\phi_2^{-1}(\pi_2) + \phi_1^{-1}(\pi_1) = \sum_{i=t-1}^{l-1} k_{i_2}(\beta_i - \beta_{i+1}) w_i + \sum_{i=0}^{t} k_{i_1}(\alpha_i - \alpha_{i+1}) v_i$, which is an integral point in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. Define $\psi(\pi) = \phi_2^{-1}(\pi_2) + \phi_1^{-1}(\pi_1)$.

On the other hand, suppose x is an integral point of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. Write $x = x_2 + x_1$ as in Eq. (4.10). Then

$$\phi_{i_2}(x_2) = \Bigg(\sigma_{l_2} \succ \cdots \succ \sigma_t \succ \sigma_{t-1};\ 0 \le c_{l_2} \le c_{l_2} + c_{l_2-1} \le \cdots \le \sum_{r=t}^{l-1} c_r \le k_{i_2}\Bigg), \tag{4.12}$$

$$\phi_{i_1}(x_1) = \Bigg(\tau_t \succ \cdots \succ \tau_2 \succ \tau_1 \succ \tau_0;\ 0 \le d_t \le d_t + d_{t-1} \le \cdots \le \sum_{r=1}^{t} d_r \le k_{i_1}\Bigg). \tag{4.13}$$

Hence, according to theorem 10.1 of [20], the path $\phi_2(x_2) * \phi_1(x_1)$ defines a path in $\Pi_\tau(\lambda)$.

DEFINITION 4.4. Let x be an integral point in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. Define $\phi(x)$ to be the LS path $f_{\alpha_{i_l}}^{e_l} \ldots f_{\alpha_{i_1}}^{e_3} f_{\alpha_{i_2}}^{e_2} f_{\alpha_{i_1}}^{e_1}(\pi_\lambda)$, where e_i are defined by $f_{\alpha_{i_l}}^{e_l} \ldots f_{\alpha_{i_1}}^{e_3} f_{\alpha_{i_2}}^{e_2} f_{\alpha_{i_1}}^{e_1}(\pi_{k_2\omega_2} * \pi_{k_1\omega_1}) = \phi_{i_2}(x_2) * \phi_{i_1}(x_1)$.

It is easy to see that $\phi \circ \psi(\pi) = \pi$. Hence, we have a bijection between LS paths in $\Pi_\tau(\lambda)$ and integral points in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$.

5. PROOF OF THEOREM 2.5

As mentioned before, we can only prove Theorem 2.5 for Lie algebras of rank two (we believe that it is also true for affine Kac–Moody algebras of rank two). We do this by finding exactly the homogeneous coordinate ring of multicone over the Schubert variety S_τ, and give an explicit flat deformation into the toric variety defined by the pair $(P_\tau(\omega_1), P_\tau(\omega_2))$. We will also show that the deformation of G/B is compatible with the deformation of $S_\tau \subset G/B$.

However, since the calculation is rather tedious, we shall give the proof only for the Schubert varieties of G_2/B (the hardest case).

This part is organized as follows. In Section 5.1, we shall study the representations $V(\omega_i) \otimes V(\omega_j)$ and $\mathrm{Sym}^2 V(\omega_i)$, since we need them to calculate the homogeneous coordinate ring of the multicone over G_2/B. In Section 5.2, we will give a Gröbner basis for the homogeneous ideal defining G/B and we show in Section 5.3 that this ideal degenerates to the homogeneous ideal defining the toric variety associated to $P_\tau(\omega_1), P_\tau(\omega_2)$ (Eqs. (A.1) and (A.2) give the vertices of these polytopes). Finally, in Sections 5.4 and 5.5, we show that this deformation is compatible with the deformation of the Schubert varieties S_τ into the appropriate toric varieties.

5.1. *Representations of G_2/B whose Highest Weight is a Fundamental Weight*

We follow the notation in chapter 22 of [9]. Since τ is maximal, the Schubert variety S_τ is equal to G/B, where $G = G_2$ and B is a Borel subgroup. The homogeneous coordinate ring of the multicone over G/B is $\bigoplus_{\lambda \text{ dominant}} H^0(G/B, \mathscr{L}_\lambda)$. It has been shown in [12] that

$$0 \to I \to \bigoplus_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \mathrm{Sym}^{k_1} V(\omega_1)^* \otimes \mathrm{Sym}^{k_2} V(\omega_2)^* \to \bigoplus_{\lambda \text{ dominant}} V(\lambda)^* \to 0, \tag{5.1}$$

where we have used $H^0(G/B, \mathscr{L}_\lambda) = V(\lambda)^*$. Moreover, I (the homogeneous ideal defining G/B) is generated by the kernel of the homomorphisms

$$\mathrm{Sym}^2 V(\omega_i)^* \to V(2\omega_i)^*, \qquad V(\omega_i)^* \otimes V(\omega_j)^* \to V(\omega_i + \omega_j)^*, \tag{5.2}$$

where $i \neq j$.

We recall that there exists an injection of $\mathfrak{g}_2$ into $\mathfrak{so}(7)$, and the representation $V(\omega_1)$ is induced from the standard representation of $\mathfrak{so}(7)$. The irreducible representation $V(\omega_2)$ is the adjoint representation of $\mathfrak{g}_2$. The weights of $V(\omega_1)$ are all the short roots, and those of $V(\omega_2)$ are all the roots plus 0. Dimension of $V(\omega_1)$ is 7 and as a vector space, it is generated by $V_4, V_3, V_1, U, W_1, W_3, W_4$, where V_4 is the highest weight vector of weight $2\alpha_1 + \alpha_2$, and the weight of the others respectively are $\alpha_1 + \alpha_2, \alpha_1, 0, -\alpha_1, -\alpha_1 - \alpha_2, -2\alpha_1 - \alpha_2$. Denote by $X_1, \ldots, X_6, H_1, H_2, Y_1, \ldots, Y_6$ a basis of $\mathfrak{g}_2$, where the weights of $X_1, \ldots, X_6$ respectively are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2$ (similarly the weights of $Y_1, \ldots, Y_6$ are $-\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2, -2\alpha_1 - \alpha_2, -3\alpha_1 - \alpha_2, -3\alpha_1 - 2\alpha_2$). Note that the H_i generate the Cartan subalgebra. We have

$$\begin{aligned} \operatorname{Sym}^2 V(\omega_1) &= V(2\omega_1) \oplus V(0), \\ \operatorname{Sym}^2 V(\omega_2) &= V(2\omega_2) \oplus V(2\omega_1) \oplus V(0), \\ V(\omega_1) \otimes V(\omega_2) &= V(\omega_1 + \omega_2) \oplus V(2\omega_1) \oplus V(\omega_1). \end{aligned} \tag{5.3}$$

Let $v_i, u, w_i \in V(\omega_1)^*$, $x_i, h_i, y_i \in V(\omega_2)^*$ be respectively the duals of $V_i, U, W_i, X_i, H_i, Y_i$. By a direct calculation we find that the trivial representation contained in $\operatorname{Sym}^2 V(\omega_1)^*$ is generated by

$$v_4 \otimes w_4 + v_3 \otimes w_3 + v_1 \otimes w_1 - u \otimes u. \tag{5.4}$$

The trivial representation contained in $\operatorname{Sym}^2 V(\omega_2)^*$ is generated by

$$\begin{aligned} &x_6 \otimes y_6 + x_5 \otimes y_5 + x_2 \otimes y_2 \\ &\quad + 3(x_4 \otimes y_4 + x_3 \otimes y_3 + x_1 \otimes y_1 + h_1 \otimes h_1 - h_1 \otimes h_2) \\ &\quad + h_2 \otimes h_2. \end{aligned} \tag{5.5}$$

On the other hand, a very long and messy calculation gives us that the representation $V(2\omega_1)^*$ contained in $\operatorname{Sym}^2 V(\omega_2)^*$ is generated by the equations of Table B.2. Continuing in this manner, the irreducible representation $V(2\omega_1)^*$ contained in $V(\omega_1)^* \otimes V(\omega_2)^*$ is generated by the equations of Table B.3. Finally, the irreducible representation $V(\omega_1)^*$ in $V(\omega_1)^* \otimes V(\omega_2)^*$ is generated by the equations of Table B.4.

5.2. *A Gröbner Basis of G_2/B (see* [11])

Applying the ideas of [10, 11] (verbatim) to our case, in this section we will find a suitable Gröbner basis of G_2/B.

We recall some notions concerning Gröbner basis (see [7] or [24]). If $\succ$ is a monomial order on the monomials of $\mathbb{C}[x_1, \ldots, x_i]$, then for any $f \in \mathbb{C}[x_1, \ldots, x_i]$, we define $in_{\succ}(f)$ to be the greatest term of f with respect to the order $\succ$; similarly, if φ is an integral weight on the monomials, then for $f = \sum c_{i_{1,\dots,i_r}} x_{i_1}^{a_1} \dots x_{i_r}^{a_r}$, the initial term $in_\varphi(f)$ is defined to be the sum of all terms $c_{i_{1,\dots,i_r}} x_{i_1}^{a_1} \dots x_{i_r}^{a_r}$, such that $\varphi(x_{i_1}^{a_1} \dots x_{i_r}^{a_r})$ is maximal. Finally, if I is an ideal, denote by $in_\varphi(I)$ the ideal generated by $in_\varphi(f)$ for all $f \in I$.

Let E_i, where $i = 1, 2$, be the basis of the representation $V(\omega_i)^*$ defined in the previous section. We consider the polynomial ring $\mathscr{R} := \mathbb{C}[\varepsilon \in E_1 \cup E_2]$. According to Eq. (5.1), the homogeneous coordinate ring of the multicone over G_2/B is $\mathscr{R}/I$.

DEFINITION 5.1. Define the integral weight function ϑ (see [7]) on the monomials of $\mathscr{R}$ as follows. First let

$$\begin{aligned}
4 &= \vartheta(v_1) = \vartheta(y_2),\\
3 &= \vartheta(v_4) = \vartheta(v_3) = \vartheta(u) = \vartheta(h_2) = \vartheta(x_1) = \vartheta(y_3),\\
2 &= \vartheta(w_1) = \vartheta(w_3) = \vartheta(x_5) = \vartheta(x_4) = \vartheta(x_3) = \vartheta(x_2)\\
&= \vartheta(h_1) = \vartheta(y_1) = \vartheta(y_4),\\
1 &= \vartheta(y_5) = \vartheta(y_6),\\
0 &= \vartheta(w_4) = \vartheta(x_6).
\end{aligned} \tag{5.6}$$

Then, define $\vartheta(\varepsilon_1 \dots \varepsilon_r) = \sum_{i=1}^{r} \vartheta(\varepsilon_i)$. We shall also define a total order $\succ$ on $E_1 \cup E_2$ as

$$\begin{aligned}
&w_4 \succ w_3 \succ y_6 \succ w_1 \succ y_5 \succ y_4 \succ y_3 \succ y_1 \succ h_1 \succ u \succ v_1 \succ y_2\\
&h_2 \succ x_1 \succ v_3 \succ x_2 \succ x_3 \succ x_4 \succ v_4 \succ x_5 \succ x_6.
\end{aligned} \tag{5.7}$$

From now on, we will write a monomial $\mathbf{m}$ of degree r in the polynomial ring $\mathscr{R}$ in the form $\varepsilon_1 \varepsilon_2 \dots \varepsilon_r$, where $\varepsilon_1 \succeq \varepsilon_2 \succeq \cdots \succeq \varepsilon_r$.

DEFINITION 5.2 (see [24, page 4]). Define $\prec_\vartheta$ to be the following (monomial) order on the set of monomials of $\mathscr{R}$: $\varepsilon_1 \dots \varepsilon_r \prec_\vartheta \varsigma_1 \dots \varsigma_s$ iff

1. $r < s$ or
2. $r = s$ and $\vartheta(\varepsilon_1 \dots \varepsilon_r) < \vartheta(\varsigma_1 \dots \varsigma_s)$ or
3. $r = s$ and $\vartheta(\varepsilon_1 \dots \varepsilon_r) = \vartheta(\varsigma_1 \dots \varsigma_s)$ and there exists an $l < r$ such that $\varepsilon_1 = \varsigma_1, \dots, \varepsilon_{l-1} = \varsigma_{l-1}$, $\varepsilon_l < \varsigma_l$.

DEFINITION 5.3. Let $\mathbf{m} = \varepsilon_1 \dots \varepsilon_r$ be a monomial in $\mathscr{R}$, where ε_i correspond to the LS path $\phi_j(\varepsilon_i) = (\tau_1^i \succ \cdots \succ \tau_{t_i}^i; 0 < a_1^i < \cdots < a_{t_i}^i = 1)$ (the correspondence is given in Table B.1). We call $\mathbf{m}$ a standard monomial if all the following hold:

1. there exists l such that $\varepsilon_1, \dots, \varepsilon_l \in E_1$ and $\varepsilon_{l+1}, \dots, \varepsilon_r \in E_2$,
2. for $i < l$ and $i > l$, we have $\tau_{t_i}^i \succeq \tau_1^{i+1}$,
3. for $i = l$, we have $\tau_{t_l}^l s_{\alpha_2} \succeq \tau_1^{l+1}$,

and if $\mathbf{m} = \varepsilon_1 \dots \varepsilon_r$ is a nonstandard monomial, then the pair $(i, i+1)$ will be called a violation of standardness iff one of the following occurs:

1. there exists i such that $\varepsilon_i \in E_2$ and $\varepsilon_{i+1} \in E_1$ or
2. $\varepsilon_i, \varepsilon_{i+1} \in E_j$ but $\tau^i_{t_i} \not\geq \tau^{i+1}_1$,
3. $\varepsilon_i \in E_1$, $\varepsilon_{i+1} \in E_2$ but $\tau^i_{t_i} s_{\alpha_2} \not\geq \tau^{i+1}_1$.

THEOREM 5.4 (see [14–16]). *Standard monomials of degree k form a basis of* $\bigoplus_{k_1+k_2=k} H^0(G/B, \mathscr{L}_{\omega_1}^{\otimes k_1} \otimes \mathscr{L}_{\omega_2}^{\otimes k_2})$ $(= \bigoplus_{k_1+k_2=k} V(k_1\omega_1 + k_2\omega_2)^*)$.

This theorem implies that if $\mathbf{n}$ is a nonstandard monomial of degree $k \geq 2$, then $\mathbf{n}$ can be written in a unique way as a linear combination of standard monomials of degree k modulo the ideal I

$$\mathbf{n} = \sum_{i=1}^{r} a_i \mathbf{s_i} \mod I, \qquad a_i \in \mathbb{C}, \tag{5.8}$$

where $\mathbf{s_i}$ are standard. Set $\mathbf{f_n} = \mathbf{n} - \sum_{i=1}^{r} a_i \mathbf{s_i}$, and

$$\mathscr{F}_2 = \{\mathbf{f_n} \mid \mathbf{n} \text{ is a nonstandard monomial of degree } 2\}. \tag{5.9}$$

In Tables B.5 and B.6, we have written down all the elements of $\mathscr{F}_2$. Looking at these tables, we observe that if $\mathbf{f}_{\varepsilon_1, \varepsilon_2} = \varepsilon_1\varepsilon_2 - \sum_{i=1}^{r} a_i \varepsilon_1^i \varepsilon_2^i$ (where $\varepsilon_1\varepsilon_2$ is a nonstandard monomial and $\varepsilon_1^i\varepsilon_2^i$ are standard) is any element of $\mathscr{F}_2$, then

1. $\varepsilon_1^i \succ \varepsilon_1$ for all $i = 1, \dots, r$ (order is given in Eq. (5.7)).
2. $in_{\prec_\vartheta}(\mathbf{f}_{\varepsilon_1, \varepsilon_2}) = \varepsilon_1\varepsilon_2$. (In Tables B.5 and B.6, we have underlined the initial term for each element.)

Now using exactly the same proofs of theorems 3.6, 3.7, and 3.8 of [11], we can show that the following holds.

LEMMA 5.5. *$\mathscr{F}_2$ is a reduced Gröbner basis for I with respect to the monomial order $\prec_\vartheta$.*

COROLLARY 5.6. *The set $\{in_\vartheta(f) \mid f \in \mathscr{F}_2\}$ is a (Gröbner) basis for $in_\vartheta(I)$.*

Proof. We have (1) $\vartheta(\varepsilon) \geq 0$ for all $\varepsilon \in E_1 \cup E_2$, and (2) $\mathscr{F}_2$ is a Gröbner basis for I with respect to $\prec_\vartheta$. Using corollary 1.9 of [24], we can prove the result.

In Tables B.5 and B.6, we have put in parentheses $in_\vartheta(f)$ for all $f \in \mathscr{F}_2$. ∎

5.3. *Flat Deformation of G_2/B into a Toric Variety*

Let $\mathscr{R}$ and I be as in the previous section. According to theorem 15.17 of [7], we have a flat family over $\mathbb{C}[t]$ of quotients of $\mathscr{R}$ whose fiber over 0 is $\mathscr{R}/in_{\vartheta}(I)$ and whose fiber over any $(t-u)$ for $u \in \mathbb{C}^*$ is $\mathscr{R}/I$. Therefore, we want to prove that $\mathscr{R}/in_{\vartheta}(I)$ is isomorphic to the homogeneous coordinate ring of the multicone over the toric variety X, defined by $P_\tau(\omega_1)$, $P_\tau(\omega_2)$ (τ-maximal).

Since τ is fixed (it is the maximal element $s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$ of the Weyl group), for simplicity we shall write $P(\omega_i) := P_\tau(\omega_i)$. As in the Introduction, let $\mathscr{B}_{k_1,k_2}$ ($k_1, k_2 \geq 0$) be the vector space generated by x^α, where α is an integral point of $k_1P(\omega_1) + k_2P(\omega_2)$, and set $\mathscr{B}$ to be the graded algebra $\oplus_{k_1,k_2}\mathscr{B}_{k_1,k_2}$, where the multiplication is $x^\alpha x^\beta = x^{\alpha+\beta}$. Then, according to [25], $\mathscr{B}$ is the homogeneous coordinate ring of the multicone over a toric variety, which we denote by X. We have

$$0 \to J \to \bigoplus_{k_1,k_2\geq 0} \operatorname{Sym}^{k_1}\mathscr{B}_{1,0} \otimes \operatorname{Sym}^{k_2}\mathscr{B}_{0,1} \to \mathscr{B} \to 0. \tag{5.10}$$

This tells us that $\mathscr{B} \simeq \mathbb{C}[p, p \in E_1' \cup E_2']/J$, where E_i' is the set of integral points of $P(\omega_i)$. Moreover, since $P(\omega_i)$'s satisfy property (1.1), J is generated by the kernel of the homomorphisms

$$\operatorname{Sym}^2\mathscr{B}_{1,0} \to \mathscr{B}_{2,0}, \qquad \operatorname{Sym}^2\mathscr{B}_{0,1} \to \mathscr{B}_{0,2}, \qquad \mathscr{B}_{1,0}\otimes\mathscr{B}_{0,1} \to \mathscr{B}_{1,1}. \tag{5.11}$$

(Compare with Eqs. (5.1), (5.2).) This equation tells us that in order to obtain the generators of J, one has to find all the relations among the integral points of $P(\omega_i)$. This is what we will do. First denote the integral points of $P(\omega_1)$ as

$$\begin{gathered}(0,0,0,0,0,0) \to \nu_4, \quad (0,0,0,0,1,0) \to \nu_3, \quad (0,0,0,1,1,0) \to \nu_1,\\ (0,0,1,1,1,0) \to \upsilon, \quad (0,0,2,1,1,0) \to \varpi_1, \quad (0,1,2,1,1,0) \to \varpi_3,\\ (1,1,2,1,1,0) \to \varpi_4.\end{gathered} \tag{5.12}$$

Similarly, we denote the integral points of $P(\omega_2)$ by

$$\begin{gathered}(0,0,0,0,0,0) \to \chi_6, \quad (0,0,0,0,0,1) \to \chi_5, \quad (0,0,0,0,1,1) \to \chi_4,\\ (0,0,0,0,2,1) \to \chi_3, \quad (0,0,0,0,3,1) \to \chi_2, \quad (0,0,0,1,2,1) \to \chi_1,\\ (0,0,1,1,2,1) \to \hbar_1, \quad (0,0,0,1,3,1) \to \hbar_2, \\ (0,0,1,1,3,1) \to \mu_1, \quad (0,0,0,2,3,1) \to \mu_2, \quad (0,0,1,2,3,1) \to \mu_3,\\ (0,0,2,2,3,1) \to \mu_4, \quad (0,0,3,2,3,1) \to \mu_5, \quad (0,1,3,2,3,1) \to \mu_6.\end{gathered} \tag{5.13}$$

By looking at the relations among these points, we obtain a basis of J. (For example, one generator of J is $\varpi_1\nu_1 - \upsilon^2$ since we have $(0,0,2,1,1,0) + (0,0,0,1,1,0) = 2(0,0,1,1,1,0)$, and so on.) In this way, we find a basis of J. This basis is given in Table B.7.

Recall that the elements in the parentheses in Tables B.5 and B.6 form a basis of $in_{\vartheta}(I)$. Note that with the following change of variables, this basis is the same as the basis of J: $v_i \to \nu_i$ for $i = 1,3,4$, $u \to \upsilon$, $w_i \to \varpi_i$ for $i = 1,3,4$, $x_i \to \chi_i$ for $i = 1,2,3,4,6$, $x_5 \to -\chi_5$, $h_i \to \hbar_i$ for $i = 1,2$, $y_i \to \mu_i$ for $i = 1,5,6$, and $y_i \to -\mu_i$ for $i = 2,3,4$. Therefore, the ring $\mathbb{C}[\varepsilon \in E_1 \cup E_2]/in_{\upsilon}(I)$ is isomorphic to the homogeneous coordinate ring of the multicone over X which is $\mathbb{C}[p \in E_1' \cup E_2']/J$. Hence we obtain the desired deformation.

5.4. *Flat Deformation of Schubert Varieties S_τ, where $\tau = s_{\alpha_j} \dots s_{\alpha_1} s_{\alpha_2}$*

The homogeneous coordinate ring of multicone over the Schubert variety S_τ can be obtained from the homogeneous coordinate ring of the multicone over G/B by setting certain variables equal to zero. Explicitly, this ring is $\mathscr{R}/I$, where the variables $\varepsilon \in E_1 \cup E_2$ are set to zero if ε is not an eigenvector of $E_\tau(\omega_i)$. In other words, the homogeneous coordinate ring of the multicone over the Schubert variety S_τ is $\mathscr{R}_\tau/I_\tau$, where $\mathscr{R}_\tau = \mathbb{C}[\varepsilon$ an eigenweight vector of $E_\tau(\omega_i)]$ and a basis of I_τ is obtained from Tables B.5 and B.6, where for $\tau = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$, set $w_4 = 0$; for $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$, set $w_4 = w_3 = y_6 = 0$; for $\tau = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$, set $w_4 = w_3 = w_1 = u = y_6 = y_5 = y_4 = y_3 = y_1 = h_1 = 0$; for $\tau = s_{\alpha_1}s_{\alpha_2}$, set everything equal to zero except $v_3, v_4, x_2, \dots, x_6$; for $\tau = s_{\alpha_2}$, set everything equal to zero except v_4, x_5, x_6.

Now, we calculate the homogeneous coordinate ring of the multicone over the toric variety X_τ defined by $P_\tau(\omega_1)$ and $P_\tau(\omega_2)$. This ring is $\mathbb{C}[p$ an integral point of $P_\tau(\omega_i)]/J_\tau$, where a basis of J_τ is obtained from the basis of J given in Table B.7 by setting equal to zero the points p which do not belong to either $P_\tau(\omega_1)$ or $P_\tau(\omega_2)$. Explicitly, for $\tau = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$, set $\varpi_4 = 0$; for $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$, set $\varpi_4 = \varpi_3 = \mu_6 = 0$; for $\tau = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}$, set $\varpi_4 = \varpi_3 = \varpi_1 = \upsilon = \mu_6 = \mu_5 = \mu_4 = \mu_3 = \mu_1 = \hbar_1 = 0$; for $\tau = s_{\alpha_1}s_{\alpha_2}$, set everything equal to zero except $\nu_3, \nu_4, \chi_2, \dots, \chi_6$; for $\tau = s_{\alpha_2}$, set everything equal to zero except ν_4, χ_5, χ_6.

It is easy to see that the same change of variables as in the previous section shows that the homogeneous coordinate ring of the multicone over X_τ is isomorphic to the ring $\mathscr{R}_\tau/in_{\vartheta}(I_\tau)$. Thus we obtain the deformation.

5.5. *Flat Deformation of Schubert Varieties S_τ, where $\tau = s_{\alpha_j} \dots s_{\alpha_2} s_{\alpha_1}$*

As mentioned in Section 5.4, we can calculate the homogeneous coordinate ring of the multicone over S_τ using the homogeneous coordinate ring of the multicone over G_2/B by setting certain variables equal to zero. Explicitly, for $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, set $y_6 = 0$; for $\tau = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, set $y_6 = y_5 = y_4 = y_3 = y_1 = h_1 = w_4 = 0$; for $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, set $y_i = h_1 = h_2 = x_1 = w_4 = w_3 = 0$, where $i = 1, \dots, 6$; for $\tau = s_{\alpha_2}s_{\alpha_1}$, set everything equal to zero except x_5, x_6, v_1, v_3, v_4; for $\tau = s_{\alpha_1}$, set everything equal to zero except x_6, v_3, v_4.

Fix an element $\tau = s_{\alpha_j} \dots s_{\alpha_2}s_{\alpha_1}$. Let $P_\tau(\omega_i)$ be as in Section 3.3 and Definition 2.3. Denote the integral points of $P_\tau(\omega_1)$ (when they belong to it) by

$$\begin{gathered}(0,0,0,0,0,0) \to \varpi_4, \quad (0,0,0,0,0,1) \to \varpi_3, \quad (0,0,0,0,1,1) \to \varpi_1,\\ (0,0,0,1,1,1) \to v, \quad (0,0,0,2,1,1) \to \nu_1, \quad (0,0,1,2,1,1) \to \nu_3,\\ (0,1,1,2,1,1) \to \nu_4. \end{gathered} \tag{5.14}$$

Similarly, we denote the integral points of $P_\tau(\omega_2)$ (when they belong to it) by

$$\begin{gathered}(0,0,0,0,0,0) \to \mu_6, \quad (0,0,0,0,1,0) \to \mu_5, \quad (0,0,0,1,1,0) \to \mu_4,\\ (0,0,0,2,1,0) \to \mu_3, \quad (0,0,0,3,1,0) \to \mu_2, \quad (0,0,1,2,1,0) \to \mu_1,\\ (0,1,1,2,1,0) \to \hbar_1, \quad (0,0,1,3,1,0) \to \hbar_2,\\ (0,1,1,3,1,0) \to \chi_1, \quad (0,0,2,3,1,0) \to \chi_2, \quad (0,1,2,3,1,0) \to \chi_3,\\ (0,2,2,3,1,0) \to \chi_4, \quad (0,3,2,3,1,0) \to \chi_5, \quad (1,3,2,3,1,0) \to \chi_6. \end{gathered} \tag{5.15}$$

Now for the homogeneous coordinate ring of the multicone over the toric variety X_τ defined by $P_\tau(\omega_1), P_\tau(\omega_2)$. This ring is $\mathbb{C}[p$ an integral point of $P_\tau(\omega_i)]/J_\tau$. A basis of J_τ is given in Table B.7, where we set equal to zero the points which do not belong to $P_\tau(\omega_1)$ or $P_\tau(\omega_2)$. Explicitly, for $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, set $\chi_6 = 0$; for $\tau = s_{\alpha_2}s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, set $\chi_6 = \chi_5 = \chi_4 = \chi_3 = \chi_1 = \hbar_1 = \nu_4 = 0$; for $\tau = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1}$, set $\chi_i = \hbar_1 = \hbar_2 = \mu_1 = \nu_4 = \nu_3 = 0$, where $i = 1, \dots, 6$; for $\tau = s_{\alpha_2}s_{\alpha_1}$, set everything equal to zero except $\mu_5, \mu_6, \nu_1, \nu_3, \nu_4$; for $\tau = s_{\alpha_1}$, set everything equal to zero except μ_6, ν_3, ν_4.

The same change of variables given at the end of Section 5.3 will do the trick.

6. APPLICATIONS

As an application of Theorem 2.4, we shall give a combinatorial description of weight multiplicities of a Demazure module.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra of rank 2 and $\lambda = k_1\omega_1 + k_2\omega_2$ a dominant weight. By the multiplicity of the weight μ in $E_\tau(\lambda)$ we mean the dimension of the vector subspace $\{v \in E_\tau(\lambda) \mid h(v) = \mu(h)v$ for $h \in \mathfrak{h}\}$. We denote this number by $m_{\lambda,\tau}(\mu)$.

According to the theorem on page 339 of [17], Char $E_\tau(\lambda) = \sum_{\pi \in \Pi_\tau(\lambda)} e^{\pi(1)}$. In other words, $m_{\lambda,\tau}(\mu)$ is the number of LS paths ending at μ. In this section, we shall show that the integral points in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$ corresponding to LS paths ending at μ, lie on a polytope with rational vertices.

Throughout this section, we fix an element τ of the Weyl group of length $\ell(\tau) = l$ and choose a reduced expression of $\tau = s_{\alpha_{i_l}} \dots s_{\alpha_{i_1}}$. Denote by $\Delta_{\lambda,\tau}$ the convex envelope of the points $\{w(\lambda) \mid w \leq \tau\}$. Let $P_\tau(\omega_i) \subset \mathbb{R}^l$ be as in Theorem 2.5.

DEFINITION 6.1. Let $v_1 = \sum_{i \text{ odd} \leq l} e_i$ and $v_2 = \sum_{i \text{ even} \leq l} e_i$, where $\{e_i\}_{i=1}^l$ is the canonical basis of $\mathbb{R}^l$. Define the linear map $\xi\colon \mathbb{R}^l \to \mathfrak{h}$ as

$$\xi(x) = -(x \cdot v_1)\alpha_{i_1} - (x \cdot v_2)\alpha_{i_2}, \tag{6.1}$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^l$.

LEMMA 6.2. *Suppose x is an integral point of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$, and let $\phi(x)$ be the LS path associated to x as in Definition* 4.4. *Then $\lambda + \xi(x) = \mu\lambda$ if and only if the LS path associated to x ends at μ.*

Proof. As before, we denote the vertices of $P_\tau(\omega_i)$ by $v_w, w \in W/W_{\omega_i}$, where v_w are defined in Definitions 2.1 and 2.3.

Let $x = \sum_{j=0}^l k(b_j v_{\tau_j})$ be an integral point of $kP_\tau(\omega_i)$ and v_i as above. Note that $x \cdot v_1 = k\sum_{r \text{ odd} \leq l} (\sum_{j=r}^l b_j)\langle \tau_r(\omega_i), \alpha_r^\vee \rangle$. Similarly, $x \cdot v_2 = k\sum_{r \text{ even} \leq l} (\sum_{j=r}^l b_j)\langle \tau_r(\omega_i), \alpha_r^\vee \rangle$. Consider the path $\phi_i(x)$ given in Eq. (3.1). Its end point is $\sum_{j=0}^l b_j \tau_j(k\omega_i)$. Explicitly,

$$\begin{aligned}\phi_i(x)(1) = k\omega_i &- \left(\sum_{r \text{ odd},\, r \leq l} \left(\sum_{j=r}^l b_j \right) \left\langle \tau_r(k\omega_i), \alpha_r^\vee \right\rangle \right) \alpha_{i_1} \\ &- \left(\sum_{r \text{ even},\, r \leq l} \left(\sum_{j=r}^l b_j \right) \left\langle \tau_r(k\omega_i), \alpha_r^\vee \right\rangle \right) \alpha_{i_2}.\end{aligned} \tag{6.2}$$

In other words, $k\omega_i + \xi(x) = \phi_i(x)(1)$.

Since x is an integral point of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$, we write $x = x_1 + x_2$ as in Proposition 4.3. Then $\phi(x) := f_{\alpha_{i_l}}^{a_{i_l}} \dots f_{\alpha_{i_1}}^{a_{i_1}}(\pi_{k_1\omega_1 + k_2\omega_2})$, such that $f_{\alpha_{i_l}}^{a_{i_l}} \dots f_{\alpha_{i_1}}^{a_{i_1}}(\pi_{k_{i_2}\omega_{i_2}} * \pi_{k_{i_1}\omega_{i_1}}) = \phi_{i_2}(x_2) * \phi_{i_1}(x_1)$. From part (a) of lemma 2.1 of [19], we have that $\phi(x)(1) = \phi_{i_2}(x_2)(1) + \phi_{i_1}(x_1)(1)$. Therefore,

$$\begin{aligned}\phi(x)(1) &= \xi(x_2) + k_{i_2}(\omega_{i_2}) + \xi(x_1) + k_{i_1}(\omega_{i_1}) \\ &= \xi(x) + (k_1\omega_1 + k_2\omega_2). \qquad (6.3)\end{aligned}$$

∎

COROLLARY 6.3. *Let λ be a dominant weight and τ an element of the Weyl group. Then we have* $\operatorname{Char} E_\tau(\lambda) = e^{\lambda}\sum e^{\xi(x)}$, *where the sum is over all the integral points x, of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$.*

APPENDIX A

A.1. *Proof of Proposition 4.3 for* $\mathfrak{g}_2$

Let the notation be as in Section 3.3. Consider the Lie algebra $\mathfrak{g}_2$ and $\tau = s_1 s_2 s_1 s_2 s_1 s_2$, which is the maximal element. Recall that the vertices of $\tilde{P}_\tau(\omega_1)$ are given by Eq. (3.5). Then according to Definition 2.3, $P_\tau(\omega_1) \subset \mathbb{R}^6$ is the convex envelope of

$$\begin{aligned}&v_{\tau_0} = (0,0,0,0,0,0), \quad v_{\tau_1} = (0,0,0,0,1,0), \quad v_{\tau_2} = (0,0,0,1,1,0),\\ &v_{\tau_3} = (0,0,2,1,1,0), \quad v_{\tau_4} = (0,1,2,1,1,0), \quad v_{\tau_5} = (1,1,2,1,1,0).\end{aligned} \qquad (A.1)$$

On the other hand, the vertices of $\tilde{P}_\tau(\omega_2)$ are specified in Eq. (3.6). So Theorem 2.4 implies that $P_\tau(\omega_2) \subset \mathbb{R}^6$ is the convex envelope of

$$\begin{aligned}&v_{\sigma_0} = (0,0,0,0,0,0), \quad v_{\sigma_1} = (0,0,0,0,0,1), \quad v_{\sigma_2} = (0,0,0,0,3,1),\\ &v_{\sigma_3} = (0,0,0,2,3,1), \quad v_{\sigma_4} = (0,0,3,2,3,1), \quad v_{\sigma_5} = (0,1,3,2,3,1)\end{aligned} \qquad (A.2)$$

For simplicity of notation, denote v_{τ_i} by w_i and v_{σ_i} by v_i. We shall first calculate the numbers r_j^i defined in Eq. (4.4):

$$\begin{aligned}&v_2 = 3w_1 + v_1, \qquad v_3 = 2w_2 + w_1 + v_1,\\ &v_4 = (3/2)w_3 + (1/2)w_2 + w_1 + v_1, \\ &v_5 = w_4 + (1/2)w_3 + (1/2)w_2 + w_1 + v_1.\end{aligned} \qquad (A.3)$$

Let x be a point in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. Then $x = \sum_{i=0}^5 a_i v_i + \sum_{i=0}^5 b_i w_i$, where $a_i \geq 0$, $b_i \geq 0$, and $\sum_{i=0}^5 a_i = k_2$, $\sum_{i=0}^5 b_i = k_1$. According to identity (A.3), we then have

$$x = b_5 w_5 + (b_4 + a_5) w_4 + \left(b_3 + \tfrac{1}{2} a_5 + \tfrac{3}{2} a_4\right) w_3 + \left(b_2 + \tfrac{1}{2} a_5 + \tfrac{1}{2} a_4 + 2a_3\right) w_2 + (b_1 + a_5 + a_4 + a_3 + 3a_2) w_1 + b_0 w_0 + (a_1 + \cdots + a_5) v_1 + a_0 v_0.$$

Therefore, we have $x = \sum_{i=0}^5 c_i w_i + (a_1 + \cdots + a_5) v_1 + a_0 v_0$. Let t be the smallest number such that $\sum_{i=t}^5 c_i \leq k_2$. Then, consider each case separately, we shall show that x_1 and x_2 satisfy the claim of Proposition 4.3.

Case 1. $b_5 \leq k_1 < b_5 + b_4 + a_5$. In this case, we have

$$x = \underbrace{b_5 w_5 + (k_1 - b_5) w_4}_{x_1} + \underbrace{A_5 v_5 + A_4 v_4 + A_3 v_3 + A_2 v_2 + A_1 v_1 + a_0 v_0}_{x_2},$$

where

$$A_5 = a_5 + (b_5 + b_4 - k_1),$$
$$A_4 = a_4 + (2/3) b_3 + 1/3(k_1 - b_5 - b_4),$$
$$A_3 = a_3 + (1/2) b_2 + 1/6(k_1 - b_5 - b_4 - b_3),$$
$$A_2 = a_2 + (1/3) b_1 + 1/6(k_1 - b_5 - b_4 - b_3 - b_2),$$
$$A_1 = a_1 + (1/3)(k_1 - b_5 - b_4 - b_3 - b_2 - b_1).$$

A close look reveals that $a_5 w_5$, $(k_1 - a_5) w_4$, $A_i v_i$ are integral points in $\mathbb{R}^6$. We have $A_5 + \cdots + A_1 + a_0 = k_2$. Hence, x_1 is an integral point of $k_1 P_\tau(\omega_1)$ and x_2 is an integral point of $k_2 P_\tau(\omega_2)$.

Case 2. $b_5 + b_4 + a_5 \leq k_1 < b_5 + b_4 + b_3 + 3/2(a_5 + a_4)$. In this case, $x = x_1 + x_2$, where

$$x_1 = b_5 w_5 + (b_4 + a_5) w_4 + (k_1 - b_5 - b_4 - a_5) w_3,$$
$$x_2 = A'_4 v_4 + A_3 v_3 + A_2 v_2 + A_1 v_1 + a_0 v_0.$$

Here $A'_4 = 2/3(b_5 + b_4 + b_3 - k_1) + a_5 + a_4$. Observe that each term is an integral point of $\mathbb{R}^6$ and that $A'_4 + A_3 + A_2 + A_1 + b_0 = k_2$. So x can be written as the sum of an integral point in $k_1 P_\tau(\omega_1)$ and an integral point in $k_2 P_\tau(\omega_2)$.

Case 3. $b_5 + b_4 + b_3 + 3/2(a_5 + a_4) \leq k_1 < b_5 + b_4 + b_3 + b_2 + 2(a_5 + a_4 + a_3)$. In this case, we could rewrite $x = x_1 + x_2$ as

$$x_1 = b_5 w_5 + (b_4 + a_5) w_4 + \left(b_3 + \tfrac{1}{2} b_5 + \tfrac{3}{2} b_4\right) w_3 + \left(k_1 - b_5 - b_4 - b_3 - \tfrac{3}{2}(a_5 + a_4)\right) w_2,$$
$$x_2 = A'_3 v_3 + A_2 v_2 + A_1 v_1 + a_0 v_0,$$

where $A'_3 = a_5 + a_4 + a_3 - 1/2(b_0 + b_1)$. Each term is an integral point and $A'_3 + A_2 + A_1 + b_0 = k_2$; we see that x_i is an integral point in $k_i P_\tau(\omega_i)$.

Case 4. $b_5 + b_4 + b_2 + 2(a_5 + a_4 + a_3) \le k_1 < b_5 + b_4 + b_3 + b_2 + b_1 + 3(a_5 + a_4 + a_3 + a_2)$. In this case, $x = x_1 + x_2$ can be written as

$$\begin{aligned} x_1 = {} & b_5 w_5 + (b_4 + a_5) w_4 + \left(b_3 + \tfrac{1}{2}a_5 + \tfrac{3}{2}a_4\right) w_3 \\ & + \left(b_2 + \tfrac{1}{2}a_5 + \tfrac{1}{2}a_4 + 2a_3\right) w_2 \\ & + \left(k_1 - b_5 - b_4 - b_3 - b_2 - 2(a_5 + a_4 + a_3)\right) w_1, \\ x_2 = {} & A'_2 v_2 + A_1 v_1 + a_0 v_0, \end{aligned}$$

where $A'_2 = a_5 + a_4 + a_3 + a_2 - (1/3)b_0$. Each term is an integral point and $A'_2 + A_1 + b_0 = k_2$. So x_i is an integral point of $k_i P_\tau(\omega_i)$.

Case 5. $b_5 + b_4 + b_3 + b_2 + b_1 + 3(a_5 + a_4 + a_3 + a_2) \le k_1$. In this case, we write $x = x_1 + x_2$ as

$$\begin{aligned} x_1 = {} & b_5 w_5 + (b_4 + a_5) w_4 + \left(b_3 + \tfrac{1}{2}a_5 + \tfrac{3}{2}a_4\right) w_3 \\ & + \left(b_2 + \tfrac{1}{2}a_5 + \tfrac{1}{2}a_4 + 2a_3\right) w_2 + (b_1 + a_5 + a_4 + a_3 + 3a_2) w_1, \\ x_1 = {} & (a_1 + \cdots + a_5) v_+ a_0 v_0. \end{aligned}$$

All the terms of the sums are integral points so that x_i is an integral point of $k_i P_\tau(\omega_i)$.

A.2. Proof of Proposition 4.3 for $A_1^{(1)}$

Consider the Kac–Moody algebra $A_1^{(1)}$ and let $\tau = s_{i_l} \dots s_1 s_2 s_1$ (we use the notation of Section 3.4), with $\ell(\tau) = l$. Then $P_\tau(\omega_1) \subset \mathbb{R}^l$ is the convex envelope of $\{v_i\}_{i=0}^{l}$ (see Eq. (3.7) and Definition 2.3),

$$v_i = (0, \dots, 0, i, i-1, \dots, 2, 1)$$

and $P_\tau(\omega_2) \subset \mathbb{R}^l$ is the convex envelope of $\{w_i\}_{i=0}^{l-1}$ (see Eq. (3.7) and Definition 2.3),

$$w_i = (0, \dots, 0, i, i-1, \dots, 2, 1, 0).$$

Step 1. We first show that $P_\tau(\omega_1)$ and $P_\tau(\omega_2)$ satisfy property (1.1). In order to do this, we write an integral point in $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$ as the sum of an integral point in $k_1 P_\tau(\omega_1)$ and an integral point in $k_2 P_\tau(\omega_2)$. The rest follows from Corollary 4.2. Observe that

$$v_j = \frac{j}{j-1} w_{j-1} + \sum_{i=1}^{j-2} \frac{1}{i(i+1)} w_i + v_1, \quad \text{where } j = 2, \dots, l. \tag{A.4}$$

Let $x = \sum_{i=0}^{l} a_i v_i + \sum_{i=0}^{l-1} b_i w_i$ (where $a_i \geq 0$, $b_i \geq 0$, $\sum a_i = k_1$, and $\sum b_i = k_2$) be a point of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$. In view of identity (A.4), we rewrite x as

$$x = \sum_{i=1}^{l-1} c_i w_i + b_0 w_0 + \left(\sum_{j=1}^{l} a_j \right) v_1 + a_0 v_0,$$

$$\text{where} \quad c_i = b_i + \frac{i+1}{i} a_{i+1} + \frac{1}{i(i+1)} \left(\sum_{j=i+2}^{l} a_j \right).$$

Now, let t be the smallest number such that $\sum_{i=t}^{l-1} c_i \leq k_2$. We then have

$$x = \sum_{i=t}^{l-1} c_i w_i + \left(k_2 - \sum_{i=t}^{l-1} c_i \right) w_{t-1} + \left(\sum_{i=t-1}^{l-1} c_i - k_2 \right) w_{t-1}$$
$$+ \sum_{i=1}^{t-2} c_i w_i + \left(\sum_{j=1}^{l} a_j \right) v_1 + a_0 v_0.$$

PROPOSITION A.1. *Let $d_t = (t-1)/t(\sum_{i=t-1}^{l-1} c_i - k_2)$ and define d_j, $2 \leq j < t$, by induction as follows*:

$$d_j = \frac{j-1}{j} \left(c_{j-1} - \frac{1}{j(j-1)} \sum_{i=j+1}^{t} d_i \right).$$

Then $d_r \geq 0$, and we have

$$x = \underbrace{\sum_{i=t}^{l-1} c_i w_i + \left(k_2 - \sum_{i=t}^{l-1} c_i \right) w_{t-1}}_{x_2} + \underbrace{\sum_{i=2}^{t} d_i v_i + \left(\frac{1}{2} b_0 + a_1 \right) v_1 + a_0 v_0}_{x_1}, \tag{A.5}$$

where x_i is a point of $k_i P_\tau(\omega_i)$ for $i = 1, 2$.

Proof. First observe that

$$\left(\sum_{i=t-1}^{l-1} c_i - k_2 \right) w_{t-1} + \sum_{i=1}^{t-2} c_i w_i + \left(\sum_{j=1}^{l} a_j \right) v_1$$
$$= \sum_{i=2}^{t} d_i \left(\frac{i}{i-1} w_{i-1} + \sum_{j=1}^{i-2} \frac{1}{j(j+1)} w_j + v_1 \right) + \left(\sum_{j=1}^{l} a_j - \sum_{j=2}^{t} d_j \right) v_1$$
$$= \sum_{i=2}^{t} d_i v_i + (1/2 b_0 + a_1) v_1,$$

where the last equality follows from Eq. (A.4). This then proves the second part of the proposition.

To prove the nonnegativity of d_r, note that according to definition of t, we have $d_t \geq 0$. On the other hand, using $\Sigma_{r=a}^{l-1} c_r = (\Sigma_{r=a}^{l-1}) b_r + \frac{a+1}{a}(\Sigma_{r=a+1}^{l} a_r)$, it follows after some long calculation that

$$d_j = \begin{cases} \frac{t-1}{t}\left(\Sigma_{i=t-1}^{l-1} b_i - k_2\right) + \Sigma_{i=t}^{l} a_i, & \text{if } j = t, \\ \frac{j-1}{j} b_{j-1} + a_j + \frac{1}{j(j+1)}\left(k_2 - \Sigma_{i=j}^{l-1} b_i\right), & \text{if } 2 \leq r < t, \end{cases}$$

and so $d_j \geq 0$ for all $2 \leq j \leq t$. That x_2 belongs to $k_2 P_{\tau_2}(\omega_2)$ is rather trivial. For x_1 use the fact that $\Sigma_{i=2}^{t} d_i = (1/2)(\Sigma_{i=1}^{l-1} b_i - k_2) + \Sigma_{i=2}^{l} a_i$. According to Eq. (A.5), x can be written as $x = x_2 + x_1$, where $x_i \in k_i P_\tau(\omega_i)$. Looking closely at the coordinates of x and x_i, it is easy to see that x is an integral point of $k_1 P_\tau(\omega_1) + k_2 P_\tau(\omega_2)$ if and only if x_i are integral points of $k_i P_\tau(\omega_i)$. We are therefore done.

APPENDIX B

We shall first write down the LS paths to which $v_j, u, w_j \in V(\omega_1)^*$, $j = 1, 3, 4$, and $x_i, h_1, h_2, y_i \in V(\omega_2)^*$, $i = 1, \ldots, 6$, correspond. Let $\tau_5 = s_1 s_2 s_1 s_2 s_1$, $\tau_4 = s_2 s_1 s_2 s_1$, $\tau_3 = s_1 s_2 s_1$, $\tau_2 = s_2 s_1$, $\tau_1 = s_1 \in W/W_{\omega_1}$, and $\sigma_5 = s_1 s_2 s_1 s_2 s_1$, $\sigma_4 = s_2 s_1 s_2 s_1$, $\sigma_3 = s_1 s_2 s_1$, $\sigma_2 = s_2 s_1$, $\sigma_1 = s_1 \in W/W_{\omega_2}$ be as in Section 2.3. Then the identities in Table B.1 hold.

In Eq. (5.3), we gave the decomposition of $\mathrm{Sym}^2 V(\omega_i)^*$, $i = 1, 2$, and $V(\omega_1)^* \otimes V(\omega_2)^*$ into irreducible representations. According to this, $V(2\omega_1)^*$ is isomorphic to a summand of $\mathrm{Sym}^2 V(\omega_i)^*$. In Table B.2, we have written down the elements which generate this representation in $\mathrm{Sym}^2 V(\omega_2)^*$. Similarly, $V(2\omega_i)^*$ and $V(\omega_1)^*$ are isomorphic to direct summands of $V(\omega_1)^* \otimes V(\omega_2)^*$. In Table B.4, we give the generators of the summand isomorphic to $V(\omega_1)^*$ and in Table B.3, the generators of the summand isomorphic to $V(2\omega_1)^*$; clearly column two specifies the weight of the corresponding generator in column three.

In Tables B.5 and B.6, we give the generators of I, the homogeneous ideal defining G/B. The number of each equation explains how it is obtained from the corresponding equations in Tables B.2, B.3, and B.4. The first generator of Table B.5, $v_4 w_4 + v_3 w_3 + v_1 w_1 - u^2$, is obtained from Eq. (5.4) and equation number (0) refers to the generator obtained from Eq. (5.5).

TABLE B.1

$v_4 \leftrightarrow (1; 0 < 1)$	$v_3 \leftrightarrow (\tau_1; 0 < 1)$
$v_1 \leftrightarrow (\tau_2; 0 < 1)$	$u \leftrightarrow (\tau_3 \succ \tau_2; 0 < 1/2 < 1)$
$w_1 \leftrightarrow (\tau_3; 0 < 1)$	$w_3 \leftrightarrow (\tau_4; 0 < 1)$
$w_4 \leftrightarrow (\tau_5; 0 < 1)$	
$x_6 \leftrightarrow (1; 0 < 1)$	$x_5 \leftrightarrow (\sigma_1; 0 < 1)$
$x_4 \leftrightarrow (\sigma_2 \succ \sigma_1; 0 < 1/3 < 1)$	$x_3 \leftrightarrow (\sigma_2 \succ \sigma_1; 0 < 2/3 < 1)$
$x_2 \leftrightarrow (\sigma_2; 0 < 1)$	$x_1 \leftrightarrow (\sigma_3 \succ \sigma_2 \succ \sigma_1; 0 < 1/2 < 2/3 < 1)$
$h_1 \leftrightarrow (\sigma_4 \succ \sigma_3 \succ \sigma_2 \succ \sigma_1; 0 < 1/2 < 2/3 < 1)$	$h_2 \leftrightarrow (\sigma_3 \succ \sigma_2; 0 < 1/2 < 1)$
$y_1 \leftrightarrow (\sigma_4 \succ \sigma_3 \succ \sigma_2; 0 < 1/3 < 1/2 < 1)$	$y_2 \leftrightarrow (\sigma_3; 0 < 1)$
$y_3 \leftrightarrow (\sigma_4 \succ \sigma_3; 0 < 1/3 < 1)$	$y_4 \leftrightarrow (\sigma_4 \succ \sigma_3; 0 < 2/3 < 1)$
$y_5 \leftrightarrow (\sigma_4; 0 < 1)$	$y_6 \leftrightarrow (\sigma_5; 0 < 1)$

TABLE B.2

	weight	eigenweight vector
(1)	$4\alpha_1 + 2\alpha_2$	$x_1 \otimes x_6 - x_3 \otimes x_5 - x_4 \otimes x_4$
(2)	$3\alpha_1 + 2\alpha_2$	$(2h_1 - h_2) \otimes x_6 + x_2 \otimes x_5 + x_3 \otimes x_4$
(3)	$2\alpha_1 + 2\alpha_2$	$y_1 \otimes x_6 - x_2 \otimes x_4 + x_3 \otimes x_3$
(4)	$3\alpha_1 + \alpha_2$	$y_2 \otimes x_6 + (h_2 - h_1) \otimes x_5 + x_1 \otimes x_4$
(5)	$2\alpha_1$	$y_3 \otimes x_5 - y_2 \otimes x_4 - x_1 \otimes x_1$
(6)	α_2	$y_5 \otimes x_6 + y_1 \otimes x_3 - h_1 \otimes x_2$
(7)	$2\alpha_1 + \alpha_2$	$2y_3 \otimes x_6 + y_1 \otimes x_5 + h_2 \otimes x_4 - x_1 \otimes x_3$
(8)	$2\alpha_1 + \alpha_2$	$y_3 \otimes x_6 + y_1 \otimes x_5 + h_1 \otimes x_4$
(9)	$\alpha_1 + \alpha_2$	$2y_4 \otimes x_6 + y_1 \otimes x_4 - h_2 \otimes x_3 + x_1 \otimes x_2$
(10)	$\alpha_1 + \alpha_2$	$y_4 \otimes x_6 + y_1 \otimes x_4 - h_1 \otimes x_3$
(11)	α_1	$y_4 \otimes x_5 - 2y_3 \otimes x_4 - y_2 \otimes x_3 - h_2 \otimes x_1$
(12)	α_1	$y_4 \otimes x_5 - y_3 \otimes x_4 - h_1 \otimes x_1$
(13)	0	$y_5 \otimes x_5 + y_4 \otimes x_4 - y_3 \otimes x_3 - y_2 \otimes x_2 + 2h_1 \otimes h_2 - h_2 \otimes h_2$
(14)	0	$y_6 \otimes x_6 - y_5 \otimes x_5 + y_3 \otimes x_3 - y_1 \otimes x_1 + 2h_1 \otimes h_2 - 3h_1 \otimes h_1$
(15)	0	$y_5 \otimes x_5 - y_4 \otimes x_4 - 2y_3 \otimes x_3 - y_1 \otimes x_1 + 2h_1 \otimes h_1 - h_1 \otimes h_2$
(16)	$-\alpha_1$	$y_5 \otimes x_4 - 2y_4 \otimes x_3 - y_3 \otimes x_2 - y_1 \otimes h_2$
(17)	$-\alpha_1$	$y_5 \otimes x_4 - y_4 \otimes x_3 - y_1 \otimes h_1$
(18)	$-\alpha_1 - \alpha_2$	$2y_6 \otimes x_4 + y_4 \otimes x_1 - y_3 \otimes h_2 + y_1 \otimes y_2$
(19)	$-\alpha_1 - \alpha_2$	$y_6 \otimes x_4 + y_4 \otimes x_1 - y_3 \otimes h_1$
(20)	$-2\alpha_1 - \alpha_2$	$2y_6 \otimes x_3 + y_5 \otimes x_1 + y_4 \otimes h_2 - y_3 \otimes y_1$
(21)	$-2\alpha_1 - \alpha_2$	$y_6 \otimes x_3 + y_5 \otimes x_1 + y_4 \otimes h_1$
(22)	$-\alpha_2$	$y_6 \otimes x_5 + y_3 \otimes x_1 - y_2 \otimes h_1$
(23)	$-2\alpha_1$	$y_5 \otimes x_3 - y_4 \otimes x_2 - y_1 \otimes y_1$
(24)	$-3\alpha_1 - \alpha_2$	$y_6 \otimes x_2 + y_5 \otimes (h_2 - h_1) + y_4 \otimes y_1$
(25)	$-2\alpha_1 - 2\alpha_2$	$y_6 \otimes x_1 - y_4 \otimes y_2 + y_3 \otimes y_3$
(26)	$-3\alpha_1 - 2\alpha_2$	$y_6 \otimes (2h_1 - h_2) + y_5 \otimes y_2 + y_4 \otimes y_3$
(27)	$-4\alpha_1 - 2\alpha_2$	$y_6 \otimes y_1 - y_5 \otimes y_3 - y_4 \otimes y_4$

TABLE B.4

	weights	eigenweight vector
(a)	$2\alpha_1 + \alpha_2$	$w_3 \otimes x_6 + w_1 \otimes x_5 - 2u \otimes x_4 - v_1 \otimes x_3 + v_3 \otimes x_1 + v_4 \otimes h_1$
(b)	$\alpha_1 + \alpha_2$	$w_4 \otimes x_6 + w_1 \otimes x_4 + 2u \otimes x_3 + v_1 \otimes x_2 + v_3 \otimes (h_1 - h_2) - v_4 \otimes y_1$
(c)	α_1	$w_4 \otimes x_5 - w_3 \otimes x_4 + 2u \otimes x_1 + v_1 \otimes (h_2 - 2h_1) + v_3 \otimes y_2 + v_4 \otimes y_3$
(d)	0	$w_4 \otimes x_4 + w_3 \otimes x_3 + w_1 \otimes x_1 + v_1 \otimes y_1 + v_3 \otimes y_3 + v_4 \otimes y_4$
(e)	$-\alpha_1$	$w_4 \otimes x_3 + w_3 \otimes x_2 + w_1 \otimes (h_2 - 2h_1) - 2u \otimes y_1 - v_3 \otimes y_4 + v_4 \otimes y_5$
(f)	$-\alpha_1 - \alpha_2$	$w_4 \otimes x_1 + w_3 \otimes (h_2 - h_1) - w_1 \otimes y_2 + 2u \otimes y_3 - v_1 \otimes y_4 - v_y \otimes y_6$
(g)	$-2\alpha_1 - \alpha_2$	$w_4 \otimes h_1 + w_3 \otimes y_1 - w_1 \otimes y_3 + 2u \otimes y_4 + v_1 \otimes y_5 + v_3 \otimes y_6$

TABLE B.3

	weight	eigenweight vector
(1′)	$4\alpha_1 + 2\alpha_2$	$v_1 \otimes x_6 - v_3 \otimes x_5 - v_4 \otimes x_4$
(2′)	$3\alpha_1 + 2\alpha_2$	$u \otimes x_6 - v_3 \otimes x_4 + v_4 \otimes x_3$
(3′)	$2\alpha_1 + 2\alpha_2$	$w_1 \otimes x_6 + v_3 \otimes x_3 - v_4 \otimes x_2$
(4′)	$3\alpha_1 + \alpha_2$	$u \otimes x_5 - v_1 \otimes x_4 + v_4 \otimes x_1$
(5′)	$2\alpha_1$	$w_3 \otimes x_5 - v_1 \otimes x_1 - v_4 \otimes y_2$
(6′)	α_2	$w_1 \otimes x_3 + u \otimes x_2 - v_3 \otimes y_1$
(7′)	$2\alpha_1 + \alpha_2$	$w_3 \otimes x_6 + u \otimes x_4 + v_1 \otimes x_3 - 2v_3 \otimes x_1 + v_4 \otimes (h_2 - h_1)$
(8′)	$2\alpha_1 + \alpha_2$	$w_1 \otimes x_5 + u \otimes x_4 + 2v_1 \otimes x_3 - v_3 \otimes x_1 + v_4 \otimes (2h_1 - h_2)$
(9′)	$\alpha_1 + \alpha_2$	$w_4 \otimes x_6 + w_1 \otimes x_4 - v_1 \otimes x_2 + v_3 \otimes (h_2 - 3h_1) + v_4 \otimes y_1$
(10′)	$\alpha_1 + \alpha_2$	$2w_1 \otimes x_4 + u \otimes x_3 - v_1 \otimes x_2 + v_3 \otimes (h_2 - 2h_1) - v_4 \otimes y_1$
(11′)	α_1	$w_4 \otimes x_5 - w_3 \otimes x_4 - v_1 \otimes h_2 - v_3 \otimes y_2 - v_4 \otimes y_3$
(12′)	α_1	$w_4 \otimes x_5 + w_3 \otimes x_4 - u \otimes x_1 - v_1 \otimes h_1 - 2v_4 \otimes y_3$
(13′)	0	$w_4 \otimes x_4 - u \otimes h_1 - v_4 \otimes y_4$
(14′)	0	$w_3 \otimes x_3 - u \otimes (h_2 - h_1) - v_3 \otimes y_3$
(15′)	0	$w_1 \otimes x_1 - u \otimes (2h_1 - h_2) - v_1 \otimes y_1$
(16′)	$-\alpha_1$	$w_4 \otimes x_3 + w_3 \otimes x_2 + w_1 \otimes h_2 + v_3 \otimes y_4 - v_4 \otimes y_5$
(17′)	$-\alpha_1$	$2w_4 \otimes x_3 + w_1 \otimes h_1 - u \otimes y_1 - v_3 \otimes y_4 - v_4 \otimes y_5$
(18′)	$-\alpha_1 - \alpha_2$	$w_4 \otimes x_1 + w_3 \otimes (h_2 - 3h_1) - w_1 \otimes y_2 + v_1 \otimes y_4 + v_4 \otimes y_6$
(19′)	$-\alpha_1 - \alpha_2$	$w_4 \otimes x_1 - w_3 \otimes (h_2 - 2h_1) + w_1 \otimes y_2 + u \otimes y_3 - 2v_1 \otimes y_4$
(20′)	$-2\alpha_1 - \alpha_2$	$w_4 \otimes (h_2 - h_1) - 2w_3 \otimes y_1 + w_1 \otimes y_3 - u \otimes y_4 + v_3 \otimes y_6$
(21′)	$-2\alpha_1 - \alpha_2$	$w_4 \otimes (2h_1 - h_2) - w_3 \otimes y_1 + 2w_1 \otimes y_3 - u \otimes y_4 + v_1 \otimes y_5$
(22′)	$-\alpha_2$	$w_3 \otimes x_1 + u \otimes y_2 - v_1 \otimes y_3$
(23′)	$-2\alpha_1$	$w_4 \otimes x_2 + w_1 \otimes y_1 - v_3 \otimes y_5$
(24′)	$-3\alpha_1 - \alpha_2$	$w_4 \otimes y_1 - w_1 \otimes y_4 - u \otimes y_5$
(25′)	$-2\alpha_1 - 2\alpha_2$	$w_4 \otimes y_2 - w_3 \otimes y_3 - v_1 \otimes y_6$
(26′)	$-3\alpha_1 - 2\alpha_2$	$w_4 \otimes y_3 - w_3 \otimes y_4 - u \otimes y_6$
(27′)	$-4\alpha_1 - 2\alpha_2$	$w_4 \otimes y_4 + w_3 \otimes y_5 - w_1 \otimes y_6$

TABLE B.5

				$w_4v_4 + w_3v_3 + (w_1v_1 - \underline{u^2})$
$\tfrac{1}{7}(0)$	$+\tfrac{1}{7}(13)$	$-\tfrac{1}{7}(14)$	$-\tfrac{3}{7}(15)$	$y_4x_4 + (y_3x_3 + \underline{y_1x_1})$
$\tfrac{10}{7}(0)$	$+\tfrac{3}{7}(13)$	$+\tfrac{18}{7}(14)$	$+\tfrac{12}{7}(15)$	$4y_6x_6 + y_5x_5 + 3y_4x_4 + 3y_3x_3 + (y_2x_2 + \underline{h_2^2})$
$\tfrac{5}{7}(0)$	$+\tfrac{5}{7}(13)$	$+\tfrac{9}{7}(14)$	$+\tfrac{6}{7}(15)$	$2y_6x_6 + y_5x_5 + 2y_4x_4 + (y_3x_3 + \underline{h_1h_2})$
$\tfrac{3}{7}(0)$	$+\tfrac{3}{7}(13)$	$+\tfrac{4}{7}(14)$	$+\tfrac{5}{7}(15)$	$y_6x_6 + y_5x_5 + (y_4x_4 + \underline{h_1^2})$
	$\tfrac{1}{4}(7')$	$+\tfrac{1}{4}(8')$	$+\tfrac{3}{4}(\mathrm{a})$	$w_3x_6 + w_1x_5 - (ux_4 - \underline{h_1v_4})$
	$\tfrac{1}{4}(7')$	$+\tfrac{1}{4}(8')$	$-\tfrac{1}{4}(\mathrm{a})$	$ux_4 + (v_1x_3 - \underline{x_1v_3})$
	$\tfrac{3}{4}(7')$	$-\tfrac{1}{4}(8')$	$+\tfrac{5}{4}(\mathrm{a})$	$2w_3x_6 + w_1x_5 - 2ux_4 - (v_1x_3 - \underline{h_2v_4})$
	$-\tfrac{1}{4}(9')$	$+\tfrac{2}{4}(10')$	$+\tfrac{1}{4}(\mathrm{b})$	$w_1x_4 + (ux_3 - \underline{y_1v_4})$
		$\tfrac{1}{2}(9')$	$+\tfrac{1}{2}(\mathrm{b})$	$w_4x_6 + w_1x_4 + (ux_3 - \underline{h_1v_3})$
	$\tfrac{3}{4}(9')$	$-\tfrac{2}{4}(10')$	$+\tfrac{5}{4}(\mathrm{b})$	$2w_4x_6 + w_1x_4 + 2ux_3 + (v_1x_2 - \underline{h_2v_3})$
	$-\tfrac{1}{4}(11')$	$+\tfrac{2}{4}(12')$	$-\tfrac{1}{4}(\mathrm{c})$	$w_3x_4 - (ux_1 + \underline{y_3v_4})$
		$\tfrac{1}{2}(11')$	$+\tfrac{1}{2}(\mathrm{c})$	$w_4x_5 - w_3x_4 + (ux_1 - \underline{h_1v_1})$
	$\tfrac{5}{4}(11')$	$-\tfrac{2}{4}(12')$	$+\tfrac{1}{4}(\mathrm{c})$	$w_4x_5 - 2w_3x_4 + ux_1 - (v_1h_2 + \underline{y_2v_3})$
	$(13')$	$+(14')$	$+(\mathrm{d})$	$2w_4x_4 + 2w_3x_3 + w_1x_1 - (uh_2 - \underline{y_1v_1})$
$\tfrac{1}{2}(13')$	$+\tfrac{3}{2}(14')$	$+\tfrac{1}{2}(15')$	$+\tfrac{1}{2}(\mathrm{d})$	$w_4x_4 + 2w_3x_3 + w_1x_1 - (uh_2 + \underline{y_3v_3})$
$-\tfrac{1}{2}(13')$	$+\tfrac{1}{2}(14')$	$+\tfrac{1}{2}(15')$	$+\tfrac{1}{2}(\mathrm{d})$	$w_3x_3 + (w_1x_1 + \underline{y_4v_4})$
$\tfrac{1}{2}(13')$	$+\tfrac{1}{2}(14')$	$+\tfrac{1}{2}(15')$	$+\tfrac{1}{2}(\mathrm{d})$	$w_4x_4 + w_3x_3 + (w_1x_1 - \underline{h_1u})$
	$\tfrac{3}{4}(16')$	$-\tfrac{2}{4}(17')$	$+\tfrac{1}{4}(\mathrm{e})$	$w_3x_2 - w_1h_1 + (w_1h_2 + \underline{y_4v_3})$
	$\tfrac{1}{4}(16')$	$+\tfrac{2}{4}(17')$	$-\tfrac{1}{4}(\mathrm{e})$	$w_4x_3 + (w_1h_1 - \underline{y_5v_4})$
		$\tfrac{1}{2}(16')$	$+\tfrac{1}{2}(\mathrm{e})$	$w_4x_3 + w_3x_2 - w_1h_1 + (w_1h_2 - \underline{y_1u})$
	$\tfrac{3}{4}(18')$	$+\tfrac{2}{4}(19')$	$-\tfrac{1}{4}(\mathrm{f})$	$w_4x_1 - (w_3h_1 - \underline{y_6v_4})$
		$\tfrac{1}{2}(18')$	$+\tfrac{1}{2}(\mathrm{f})$	$w_4x_1 + w_3h_2 - 2w_3h_1 - (w_1y_2 - \underline{y_3u})$
	$\tfrac{1}{4}(18')$	$-\tfrac{2}{4}(19')$	$+\tfrac{1}{4}(\mathrm{f})$	$w_3h_2 - 2w_3h_1 - (w_1y_2 - \underline{y_4v_1})$
	$\tfrac{3}{4}(20')$	$-\tfrac{1}{4}(21')$	$+\tfrac{1}{4}(\mathrm{g})$	$w_4h_2 - w_4h_1 - (w_3y_1 - \underline{y_6v_3})$
	$\tfrac{1}{4}(20')$	$-\tfrac{3}{4}(21')$	$-\tfrac{1}{4}(\mathrm{g})$	$w_4h_2 - 2w_4h_1 - (w_1y_3 + \underline{y_5v_1})$
	$-\tfrac{1}{4}(20')$	$-\tfrac{1}{4}(21')$	$\tfrac{1}{4}(\mathrm{g})$	$w_3y_1 - (w_1y_3 - \underline{y_4u})$

TABLE B.6

(1)	$x_1x_6 - (x_3x_5 + \underline{x_4^2})$	(16)	$y_5x_4 - 2y_4x_3 - (y_3x_2 + \underline{y_1h_2})$
(2)	$2h_1x_6 - h_2x_6 + (x_2x_5 + \underline{x_3x_4})$	(17)	$y_5x_4 - (y_4x_3 + \underline{y_1h_1})$
(3)	$y_1x_6 - (x_2x_4 - \underline{x_3^2})$	(18)	$2y_6x_4 + y_4x_1 - (y_3h_2 - \underline{y_1y_2})$
(4)	$y_2x_6 - h_1x_5 + (h_2x_5 + \underline{x_1x_4})$	(19)	$y_6x_4 + (y_4x_1 - \underline{y_3h_1})$
(5)	$y_3x_5 - (y_2x_4 + \underline{x_1^2})$	(20)	$2y_6x_3 + y_5x_1 + (y_4h_2 - \underline{y_3y_1})$
(6)	$y_5x_6 + (y_1x_3 - \underline{h_1x_2})$	(21)	$y_6x_3 + (y_5x_1 + \underline{y_4h_1})$
(7)	$2y_3x_6 + y_1x_5 + (h_2x_4 - \underline{x_1x_3})$	(22)	$y_6x_5 + (y_3x_1 - \underline{y_2h_1})$
(8)	$y_3x_6 + (y_1x_5 + \underline{h_1x_4})$	(23)	$y_5x_3 - (y_4x_2 + \underline{y_1^2})$
(9)	$2y_4x_6 + y_1x_4 - (h_2x_3 - \underline{x_1x_2})$	(24)	$y_6x_2 - y_5h_1 + (y_5h_2 + \underline{y_4y_1})$
(10)	$y_4x_6 + (y_1x_4 - \underline{h_1x_3})$	(25)	$y_6x_1 - (y_4y_2 - \underline{y_3^2})$
(11)	$y_4x_5 - 2y_3x_4 - (y_2x_3 + \underline{h_2x_1})$	(26)	$2y_6h_1 - y_6h_2 + (y_5y_2 + \underline{y_4y_3})$
(12)	$y_4x_5 - (y_3x_4 + \underline{h_1x_1})$	(27)	$y_6y_1 - (y_5y_3 + \underline{y_4^2})$
(1′)	$\upsilon_1x_6 - (\upsilon_3x_5 + \underline{x_4\upsilon_4})$	(22′)	$w_3x_1 + (uy_2 - \underline{y_3\upsilon_1})$
(2′)	$ux_6 - (\upsilon_3x_4 - \underline{x_3\upsilon_4})$	(23′)	$w_4x_2 + (w_1y_1 - \underline{y_5\upsilon_3})$
(3′)	$w_1x_6 + (\upsilon_3x_3 - \underline{x_2\upsilon_4})$	(24′)	$w_4y_1 - (w_1y_4 + \underline{y_5u})$
(4′)	$ux_5 - (\upsilon_1x_4 - \underline{x_1\upsilon_4})$	(25′)	$w_4y_2 - (w_3y_3 + \underline{y_6\upsilon_1})$
(5′)	$w_3x_5 - (\upsilon_1x_1 + \underline{y_2\upsilon_4})$	(26′)	$w_4y_3 - (w_3y_4 + \underline{y_6u})$
(6′)	$w_1x_3 + (ux_2 - \underline{y_1\upsilon_3})$	(27′)	$w_4y_4 + (w_3y_5 - \underline{y_6w_1})$

TABLE B.7

$\nu_1\varpi_1 - \upsilon^2$	$\mu_3\chi_2 - \mu_1\hbar_2$	$\upsilon\chi_4 - \hbar_1\nu_4$	$\varpi_3\hbar_1 - \mu_6\nu_4$
$\chi_3\chi_5 - \chi_4^2$	$\mu_4\chi_3 - \mu_1\hbar_1$	$\nu_1\chi_3 - \chi_1\nu_3$	$\varpi_1\mu_2 - \mu_3\upsilon$
$\chi_2\chi_5 - \chi_3\chi_4$	$\mu_3\hbar_2 - \mu_1\mu_2$	$\nu_1\chi_3 - \hbar_2\nu_4$	$\varpi_1\mu_2 - \mu_4\nu_1$
$\chi_2\chi_4 - \chi_3^2$	$\mu_4\chi_1 - \mu_3\hbar_1$	$\upsilon\chi_3 - \mu_1\nu_4$	$\varpi_3\mu_1 - \mu_6\nu_3$
$\hbar_2\chi_5 - \chi_1\chi_4$	$\mu_4\hbar_2 - \mu_3\mu_1$	$\upsilon\chi_3 - \hbar_1\nu_3$	$\varpi_1\mu_3 - \mu_5\nu_1$
$\mu_2\chi_4 - \chi_1^2$	$\mu_5\chi_1 - \mu_4\hbar_1$	$\nu_1\chi_2 - \hbar_2\nu_3$	$\varpi_1\nu_3 - \nu_4\upsilon$
$\mu_1\chi_3 - \hbar_1\chi_2$	$\mu_3\chi_1 - \mu_2\hbar_1$	$\upsilon\chi_1 - \mu_3\nu_4$	$\upsilon\mu_2 - \mu_3\nu_1$
$\hbar_2\chi_4 - \chi_1\chi_3$	$\mu_4\chi_2 - \mu_1^2$	$\upsilon\chi_1 - \hbar_1\nu_1$	$\varpi_1\mu_1 - \mu_5\nu_3$
$\mu_1\chi_5 - \hbar_1\chi_4$	$\mu_5\hbar_2 - \mu_4\mu_1$	$\nu_1\hbar_2 - \mu_2\nu_3$	$\varpi_1\mu_4 - \mu_5\upsilon$
$\hbar_2\chi_3 - \chi_1\chi_2$	$\mu_4\mu_2 - \mu_3^2$	$\upsilon\hbar_2 - \mu_1\nu_1$	$\varpi_3\mu_3 - \mu_6\nu_1$
$\mu_1\chi_4 - \hbar_1\chi_3$	$\mu_5\mu_2 - \mu_4\mu_3$	$\upsilon\hbar_2 - \mu_3\nu_3$	$\varpi_3\mu_4 - \mu_6\upsilon$
$\mu_2\chi_3 - \hbar_2\chi_1$	$\mu_5\mu_3 - \mu_4^2$	$\varpi_1\chi_1 - \mu_4\nu_4$	$\mu_5\varpi_3 - \mu_6\varpi_1$
$\mu_3\chi_4 - \hbar_1\chi_1$	$\nu_3\chi_5 - \chi_4\nu_4$	$\varpi_1\chi_1 - \hbar_1\upsilon$	
$\mu_3\chi_3 - \mu_1\chi_1$	$\nu_3\chi_4 - \chi_3\nu_4$	$\varpi_1\hbar_2 - \mu_4\nu_3$	
$\mu_2\chi_2 - \hbar_2^2$	$\nu_3\chi_3 - \chi_2\nu_4$	$\varpi_1\hbar_1 - \mu_5\nu_4$	
$\mu_3\chi_3 - \hbar_2\hbar_1$	$\nu_1\chi_4 - \chi_1\nu_4$	$\varpi_1\hbar_2 - \mu_1\upsilon$	
$\chi_4\mu_4 - \hbar_1^2$	$\nu_1\chi_1 - \mu_2\nu_4$		
	$\upsilon\chi_2 - \mu_1\nu_3$		

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